

# The Goldbach Conjecture

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## Abstract

Let  $a \geq 1682$  be an even number such that  $a$  is not twice a prime. Let  $\{p_1, p_2, p_3, \dots, p_k\}$  be the ordered set of  $k$  primes less than  $\sqrt{a}$ . Every natural number  $n < a$  is associated with a  $k$ -tuple, the elements of which are the remainders of dividing  $n$  by  $p_1, p_2, p_3, \dots, p_k$ . Consequently, we have a sequence of  $k$ -tuples of remainders. Besides, we prove that if  $p$  is a prime number less than  $a$ , then if  $p \not\equiv a \pmod{p_h}$  for every  $p_h \in \{p_1, p_2, p_3, \dots, p_k\}$ , then  $a - p$  is a prime or  $a - p = 1$ . If a given  $k$ -tuple has a remainder 0 or has a remainder equal to the remainder of dividing  $a$  by  $p_h$ , for any  $p_h \in \{p_1, p_2, p_3, \dots, p_k\}$ , we say that it is a *prohibited*  $k$ -tuple, and otherwise we say that it is a *permitted*  $k$ -tuple. If we make the sequence of  $k$ -tuples and select the appropriate remainders for the given number  $a$ , the indices of the *permitted*  $k$ -tuples within the interval  $[1, a]$  are primes  $p$  such that  $a - p$  is a prime or  $a - p = 1$ . We prove that for  $a \geq 1682$ , within the interval  $[1, a]$  there are 3 or more *permitted*  $k$ -tuples. This is sufficient to prove the Goldbach Conjecture for every even number  $a \geq 1682$ .

# Contents

<b>1</b>	<b>Introduction</b>	<b>3</b>
<b>2</b>	<b>Periodic sequences of <math>k</math>-tuples</b>	<b>5</b>
<b>3</b>	<b>The density of <i>permitted</i> <math>k</math>-tuples</b>	<b>10</b>
<b>4</b>	<b>The average number of <i>permitted</i> <math>k</math>-tuples within a given interval <math>\mathbb{I}[x, y]</math></b>	<b>13</b>
<b>5</b>	<b>The density of <i>permitted</i> <math>k</math>-tuples within the interval <math>\mathbb{I}[1, y]</math> for <math>y \rightarrow \infty</math></b>	<b>16</b>
<b>6</b>	<b>The number of <i>permitted</i> <math>k</math>-tuples within the interval <math>\mathbb{I}[1, p_k^2]</math></b>	<b>18</b>
6.1	The <i>true</i> density $\delta_k^{\mathbb{I}}$ and the <i>selected</i> remainders . . . . .	19
6.2	Probabilistic Model for the <i>true</i> density $\delta_k^{\mathbb{I}}$ . . . . .	20
6.3	Hypergeometric Model . . . . .	21
6.4	A Lower Confidence Limit for $Y_k^B$ . . . . .	23
6.5	Comparison of the Models . . . . .	24
6.6	Calculating the number of <i>permitted</i> $k$ -tuples within $\mathbb{I}[1, p_k^2]$ . . . . .	24
<b>7</b>	<b>The Fundamental Theorem and the <i>permitted</i> <math>k</math>-tuples</b>	<b>28</b>

# Chapter 1

## Introduction

In the year 1742 Goldbach wrote a letter to his friend Euler telling him about a conjecture involving prime numbers. *Goldbach conjecture: Every even number  $a > 4$  is the sum of two odd primes*. The Goldbach Conjecture is one of the oldest unsolved problems in number theory [6]. This conjecture was verified many times with powerful computers, but until the present time it could not be proved. Until March 30, 2012, T. Oliveira e Silva has verified the conjecture for  $n \leq 36 \times 10^{17}$  [11]. Mathematicians have achieved some results in their efforts to prove this conjecture. Vinogradov proved, in 1937, that every sufficiently large odd number is the sum of three primes [2]. Later on, J.R. Chen showed in 1973, that every sufficiently large even number can be written as the sum of either two primes, or a prime and the product of two primes [3]. In 1975, H. Montgomery and R.C. Vaughan showed that "most" even numbers were expressible as the sum of two primes [7]. In this work we prove the following:

**Proposition 1.1.** *The Main Proposition*

*Every even number  $a \geq 1682$  is the sum of two odd primes.*

If  $a \geq 1682$  is twice a prime the proposition is true. Therefore, to prove the Main Proposition it is sufficient to prove for  $a \geq 1682$ , where  $a$  is not twice a prime, that we have at least 2 primes such that the sum is  $a$ .

Let  $\{p_1, p_2, p_3, \dots, p_k\}$  be the ordered set of  $k$  primes less than  $\sqrt{a}$ . We associate every natural number  $n$  with a  $k$ -tuple, the elements of which are the remainders of dividing  $n$  by  $p_1, p_2, p_3, \dots, p_k$ , and consequently we have a periodic sequence of  $k$ -tuples of remainders. If we arrange the  $k$ -tuples from top to bottom, the sequence of  $k$ -tuples of remainders can be seen as a matrix made of  $k$  columns and infinite rows, where each column is a periodic sequence, denoted by  $s_h$  ( $1 \leq h \leq k$ ), of the remainders  $n \pmod{p_h}$ . The sequence of  $k$ -tuples is denoted by the symbol  $S_k^a$ , and is called the partial sum  $S_k^a$ . We are really interested in the interval  $[1, a]$  of the sequence  $S_k^a$ .

In the  $k$ -tuples of the sequence  $S_k^a$ , if there are any remainder 0 or any of the remainders of dividing  $a$  by  $p_1, p_2, p_3, \dots, p_k$ , these remainders are defined as *selected* remainders. Therefore, within the periods of every sequence  $s_h$  we have that 0 is ever a *selected* remainder and besides, if  $p_h$  does not divide  $a$ , the remainder is also a *selected* remainder. If a given  $k$ -tuple has one or more *selected* remainders, we say that it is a *prohibited*  $k$ -tuple, and otherwise we say that it is a *permitted*  $k$ -tuple. We prove that the indices of the *permitted*  $k$ -tuples within the interval  $[1, a]$  are primes  $p$  such that  $a - p$  is a prime or  $a - p = 1$ . In other words, the indices of the *permitted*  $k$ -tuples where  $a - p \neq 1$  are Goldbach partitions [11] for the even number  $a$ . For proving our Main Proposition it is sufficient to prove that for every even number  $a$ , there are 3 or more *permitted*  $k$ -tuples within the interval  $[1, a]$  (Note that the index 1 could be associated to a permitted  $k$ -tuple).

It must be taking into account that if  $a$  is divisible by any of the primes  $p_1, p_2, p_3, \dots, p_k$  the remainder is 0, and therefore, in the sequences  $s_h$  that make the partial sum  $S_k^a$  could exist one or two selected remainders within the period of size  $p_h$  (If there is only one selected remainder it is ever 0). Therefore, we consider the intervals  $[1, p_k^2] \subset [1, a]$  within the more general partial sums denoted by  $S_k$ , where in the sequences  $s_h$ , by definition, there are two selected remainders within the periods of size  $p_h$ , and each of them could be or not could be equal to 0. This way, proving that from some level  $k$  onward, within the interval  $[1, p_k^2]$  of the partial sum  $S_k$  there are 3 or more permitted  $k$ -tuples, the same is true for the interval  $[1, a]$  of the partial sum  $S_k^a$ .

Suppose that we have a given partial sum  $S_k$  (In other words, a given sequence of  $k$ -tuples  $S_k$ ), where  $p_k$  denotes the greatest prime (Characteristic prime of  $S_k$ ) and  $m_k$  denotes the period. The number of permitted  $k$ -tuples within the period, that is to say within the interval  $[1, m_k]$ , denoted by  $c_k$ , can be computed and not depends on what are the selected remainders within the sequences  $s_h$  that make the partial sum  $S_k$ . When we increase the level  $k$ , it is easy to see that  $c_k$  grows up, but the quotient  $c_k/m_k$  tends to 0. However, if we count the average number of permitted  $k$ -tuples within subintervals of size  $p_k$  within a period of  $S_k$ , we prove that this quantity, that we denote  $\delta_k$ , grows up and tends to  $\infty$ .

Since we are interested in the number of permitted  $k$ -tuples within intervals of  $S_k$ , we denote by  $\delta_k^I$  the density of permitted  $k$ -tuples within a given interval  $[1, y]$  ( $1 \leq y < m_k$ ) of  $S_k$ . In other words, for the interval  $[1, y]$ , we denote by  $\delta_k^I$  the average number of permitted  $k$ -tuples within subintervals of size  $p_k$ . While the density  $\delta_k$  of permitted  $k$ -tuples within a period does not change if we choose another selected remainders within the sequences  $s_h$ , the density

$\delta_k^I$  of permitted  $k$ -tuples within the interval  $[1, y]$  does depend on the combination of selected remainders within the sequences  $s_h$  that make  $S_k$ . In the Chapter 4 we prove that the average of  $\delta_k^I$  for all the combinations of selected remainders coincide with the density  $\delta_k$  within the period of  $S_k$ .

Now, let us consider, for the partial sum  $S_k$ , the interval  $[1, y = p_k^2]$ . For each of the partial sums  $S_h$  from level  $h = 1$  to level  $h = k$  we have the interval  $[1, y = p_k^2]_h$  either. We prove that, for a given level  $h$ , as the size  $y$  of the interval tends to  $\infty$ , we have  $\delta_4^I \rightarrow \delta_4$ . In other words, if we stay at level  $h = 4$ , for  $k$  large enough the interval  $[1, y = p_k^2]_4$  will be many times greater than the period  $m_{h=4}$  of the partial sum  $S_{h=4}$  and then the density  $\delta_4^I$  will be more and more close to the average density  $\delta_4$ . Consequently, for a given  $\epsilon$  small enough and for level  $k$  large enough onward we have the *true* density  $\delta_4^I$  within the bounds  $(\delta_4 - \epsilon)$  and  $(\delta_4 + \epsilon)$ .

Since the average density  $\delta_k$  grows up and tends to  $\infty$  we assume that for a level  $k$  large enough onward the increment of  $\delta_k^I$  from level  $h = 4$  to level  $h = k$  should be greater than  $\epsilon$  and then it must be  $\delta_k^I > \delta_4 + \epsilon$ . To find this level  $k$  we define a probabilistic model and confidence limits for  $\delta_k^I$  and we prove that  $\delta_k^I > 0.430$  for level  $k \geq 13$ . This result implies that for level  $k \geq 13$  the number of permitted  $k$ -tuples within the interval  $[1, y = p_k^2]$  is greater than or equal to 3, as desired.

Briefly, the plan of the proof is the following:

- We define the sequences of  $k$ -tuples of remainders of dividing  $n$  by  $p_1, p_2, p_3, \dots, p_k$ , denoted by  $S_k$ , and we study its properties (Chapter 2).
- We define the concepts of *selected* remainders, *permitted*  $k$ -tuples and *prohibited*  $k$ -tuples.
- We define the concept of density of *permitted*  $k$ -tuples within a given interval of  $S_k$  (Chapter 3).
- We prove that the density of *permitted*  $k$ -tuples within a period of  $S_k$ , denoted by  $\delta_k$ , grows up and tends to  $\infty$ .
- We prove that the average density of *permitted*  $k$ -tuples within the interval  $[1, y]$  of  $S_k$  is equal to the density of *permitted*  $k$ -tuples within the period of  $S_k$  (Chapter 4).
- For a given level  $k$  of  $S_k$ , we prove that the density of *permitted*  $k$ -tuples within the interval  $[1, y]$ , denoted by  $\delta_k^I$ , tends to the average  $\delta_k$  as the size  $y$  of the interval tends to  $\infty$  (Chapter 5).
- We define a probabilistic model and confidence limits for  $\delta_k^I$  and we prove that  $\delta_k^I > 0.430$  for level  $k \geq 13$ . This result implies that for level  $k \geq 13$  the number of permitted  $k$ -tuples within the interval  $[1, y = p_k^2]$  is greater than or equal to 3 (Chapter 6).
- We define the sequence of  $k$ -tuples associated to  $a$ , denoted by  $S_k^a$ , where if a given  $k$ -tuple has a remainder 0 or has a remainder equal to the remainder of dividing  $a$  by  $p_h$ , for any  $p_h \in \{p_1, p_2, p_3, \dots, p_k\}$ , it is a *prohibited*  $k$ -tuple. Otherwise it is a *permitted*  $k$ -tuple (Chapter 7).
- We prove that if  $p < a$  is a prime, then if besides  $p \not\equiv a \pmod{p_h}$  for every  $p_h \in \{p_1, p_2, p_3, \dots, p_k\}$ , then  $a - p = 1$  or  $a - p$  is also a prime (Fundamental Theorem).
- Using the preceding proposition, we prove that the indice  $n$  of a given *permitted*  $k$ -tuple ( $1 < n < a$ ) of  $S_k^a$  is a prime  $p = n$  such that  $a - p = 1$  or  $a - p$  is also a prime.
- We prove that in the sequence  $S_k^a$ , for level  $k \geq 13$ , the number of *permitted*  $k$ -tuples within intervals of size  $[1, a]$  is greater than or equal to 3.
- Finally we prove that at least one of these *permitted*  $k$ -tuples in the interval  $[1, a]$  is associated to a prime  $p$  such that  $a - p$  is also a prime.

## Chapter 2

# Periodic sequences of $k$ -tuples

We begin introducing the concept of sequence of remainders and the related concept of sequence of  $k$ -tuples of remainders.

**Definition 2.1.** Given a prime number  $p$ , we define the periodic sequence  $\{r_n\}$ , where  $r_n$  denotes the remainder of dividing  $n$  by the modulo  $p$ . We denote the sequence  $\{r_n\}$  by the symbol  $s$ . The period of the sequence is equal to  $p$ .

**Example 2.1.** For modulo  $p = 5$  we have  $s = \{1, 2, 3, 4, 0, 1, 2, 3, \dots\}$

**Definition 2.2.** Given a natural number  $n$  and any set of  $k$  primes  $\{p_1, p_2, p_3, \dots, p_k\}$ , we define the  $k$ -tuple  $(r_1 \ r_2 \ r_3 \dots r_k)$ , where  $r_1, r_2, r_3, \dots, r_k$  denote the remainders of dividing  $n$  by the modulus  $p_1, p_2, p_3, \dots, p_k$ .

**Definition 2.3.** Given any set of  $k$  primes  $\{p_1, p_2, p_3, \dots, p_k\}$ , we define the sequence  $\{(r_1 \ r_2 \ r_3 \dots r_k)_n\}$ , the elements of which are  $k$ -tuples of remainders obtained dividing  $n$  by the modulus  $p_1, p_2, p_3, \dots, p_k$ . The sequence of  $k$ -tuples of remainders can be seen as a matrix made of  $k$  columns and infinite rows. Note that each column of this matrix is a periodic sequence of period  $p_h$  ( $1 \leq h \leq k$ ), and it will also be called a sequence of unary tuples with period or modulo  $p_h$ .

**Definition 2.4.** The formal addition of sequences operation  $(+)$ .

Let  $\{p_1, p_2, p_3, \dots, p_j\}$  and  $\{q_1, q_2, q_3, \dots, q_k\}$  be two disjoint sets of primes. Let  $\{(a_1 \ a_2 \ a_3 \dots a_j)_n\}$  be the sequence of  $j$ -tuples of the remainders of dividing  $n$  by the  $j$  prime modulus  $\{p_1, p_2, p_3, \dots, p_j\}$  and let  $\{(b_1 \ b_2 \ b_3 \dots b_k)_n\}$  be the sequence of  $k$ -tuples of the remainders of dividing  $n$  by the  $k$  prime modulus  $\{q_1, q_2, q_3, \dots, q_k\}$ . We define the sum of the sequences to be the sequence of  $(j+k)$ -tuples made by the ordered juxtaposition of each  $j$ -tuple of the first sequence with each  $k$ -tuple of the second sequence:  $\{(a_1 \ a_2 \ a_3 \dots a_j)_n\} + \{(b_1 \ b_2 \ b_3 \dots b_k)_n\} = \{(a_1 \ a_2 \ a_3 \dots a_j \ b_1 \ b_2 \ b_3 \dots b_k)_n\}$ .

**Example 2.2.** Table 2.1 shows the sequence of 3-tuples of remainders of dividing  $n$  by  $\{2, 3, 5\}$ , the sequence of 2-tuples of remainders of dividing  $n$  by  $\{7, 11\}$  and the formal sum of both sequences.

**Definition 2.5.** Let  $\{p_1, p_2, p_3, \dots, p_k, \dots\}$  be the sequence of primes. We denote by the symbol  $s_k$  the sequence  $\{r_n\}$  of the remainders of dividing  $n$  by the modulo  $p_k$ . Let us consider a sequence of sequences  $\{s_k\}$ . Then, we define the series denoted by  $\sum s_k$  to be the sequence  $\{S_k\}$  where  $S_k$  express the partial sums:

$$\begin{array}{rcl} S_1 & = & s_1 \\ S_2 & = & s_1 + s_2 \\ S_3 & = & s_1 + s_2 + s_3 \\ S_4 & = & s_1 + s_2 + s_3 + s_4 \\ \vdots & & \vdots \\ S_k & = & s_1 + s_2 + s_3 + s_4 + \dots + s_k \end{array}$$

and the symbol  $\sum$  refers to the formal addition of sequences. In each partial sum  $S_k$ , the greatest prime modulo  $p_k$  will be called characteristic prime modulo of the partial sum  $S_k$ . The index  $k$  will be called level  $k$ , and we shall say that  $S_k$  is the partial sum of level  $k$ .

**Example 2.3.** Table 2.2 shows the partial sum  $S_k = S_4$  and the formal addition of the sequence of unary tuples  $s_k = s_5$  to obtain the partial sum  $S_{k+1} = S_5$ .

**Remark 2.1.** A given partial sum  $S_k$  is a sequence where the elements are  $k$ -tuples of remainders obtained dividing  $n$  by the modulus  $p_1, p_2, p_3, \dots, p_k$ .

Table 2.1: Formal sum of sequences

$n$	2	3	5	7	11	2	3	5	7	11
1	1	1	1	1	1	1	1	1	1	1
2	0	2	2	2	2	0	2	2	2	2
3	1	0	3	3	3	1	0	3	3	3
4	0	1	4	4	4	0	1	4	4	4
5	1	2	0	5	5	1	2	0	5	5
6	0	0	1	6	6	0	0	1	6	6
7	1	1	2	0	7	1	1	2	0	7
8	0	2	3	1	8	0	2	3	1	8
9	1	0	4	+	2	9	=	1	0	4
10	0	1	0	3	10	0	1	0	3	10
11	1	2	1	4	0	1	2	1	4	0
12	0	0	2	5	1	0	0	2	5	1
13	1	1	3	6	2	1	1	3	6	2
14	0	2	4	0	3	0	2	4	0	3
15	1	0	0	1	4	1	0	0	1	4
16	0	1	1	2	5	0	1	1	2	5
17	1	2	2	3	6	1	2	2	3	6
18	0	0	3	4	7	0	0	3	4	7
.	.	.	.	.	.	.	.	.	.	.

**Definition 2.6.** Given a sequence of unary tuples  $\{r_n\}$  with prime modulo  $p_k$  we assign to the remainders  $r_n$  one of the two following states: *selected* state - *not selected* state.

**Definition 2.7.** Given a partial sum  $S_k$ , we define one  $k$ -tuple to be *prohibited* if it has one or more *selected* remainders, and we define it to be *permitted* if it has not any *selected* remainder.

From now on we denote by  $\sum s_k$  or  $\{S_k\}$  a given series where the following restrictions are applied to the sequences of unary tuples  $s_h$  ( $1 \leq h \leq k$ ) that make the partial sum  $S_k$ :

**Definition 2.8.** Let  $s_h$  ( $1 \leq h \leq k$ ) be one of the sequences of unary tuples that make the partial sum  $S_k$ . We define:

Restriction 1: If  $h = 1$ , in the sequence of unary tuples  $s_1$  will be *selected* one remainder, the same in every period of the sequence.

Restriction 2: If  $1 < h \leq k$ , on each sequence of unary tuples  $s_h$  will be *selected* two remainders, the same two in every period of the sequence.

**Example 2.4.** Table 2.3 shows the sequence of 4-tuples of remainders modulus  $\{2, 3, 5, 7\}$  where it can be seen marked between  $[]$  the *selected* remainders. Note that the 4-tuples 1 and 7 are *permitted*  $k$ -tuples.

**Definition 2.9.** We denote by  $m_k$  the product  $p_1 p_2 p_3 \dots p_k$ .

**Proposition 2.1.** The partial sum  $S_k$  is a periodic sequence of  $k$ -tuples and the period is equal to  $m_k = p_1 p_2 p_3 \dots p_k$ .

*Proof.* Let  $s_h$  ( $1 \leq h \leq k$ ) be the sequences of unary tuples that make the partial sum  $S_k$ . Let  $m'$  be a given multiple of all the primes  $p_1, p_2, p_3, \dots, p_k$ . The period of every sequence  $s_h$  is equal to  $p_h \in \{p_1, p_2, p_3, \dots, p_k\}$ . Therefore, for every sequence  $s_h$  the remainders are repeated for all the intervals of size  $m'$ , starting from the index  $n = 1$  onward. Since  $p_1, p_2, p_3, \dots, p_k$  are primes we have that  $m_k$  is the Least Common Multiple. Consequently, the period of  $S_k$  is equal to  $m_k$ . ■

**Definition 2.10.** For a given partial sum  $S_k$  we denote by  $c_k$  the number of *permitted*  $k$ -tuples within a period of  $S_k$ .

**Proposition 2.2.** Let  $S_k$  be a given partial sum. We have:  $c_k = (p_1 - 1) (p_2 - 2) (p_3 - 2) \dots (p_k - 2)$ .

Table 2.2: Partial sums  $S_4$  and  $S_5$ 

$n$	$S_4$				$s_5$	$S_5$				
	2	3	5	7		2	3	5	7	11
1	1	1	1	1	1	1	1	1	1	1
2	0	2	2	2	2	0	2	2	2	2
3	1	0	3	3	3	1	0	3	3	3
4	0	1	4	4	4	0	1	4	4	4
5	1	2	0	5	5	1	2	0	5	5
6	0	0	1	6	6	0	0	1	6	6
7	1	1	2	0	7	1	1	2	0	7
8	0	2	3	1	8	0	2	3	1	8
9	1	0	4	2	9	1	0	4	2	9
10	0	1	0	3	10	0	1	0	3	10
11	1	2	1	4	0	1	2	1	4	0
12	0	0	2	5	1	0	0	2	5	1
13	1	1	3	6	2	1	1	3	6	2
14	0	2	4	0	3	0	2	4	0	3
15	1	0	0	1	4	1	0	0	1	4
16	0	1	1	2	5	0	1	1	2	5
17	1	2	2	3	6	1	2	2	3	6
18	0	0	3	4	7	0	0	3	4	7
.	.	.	.	.	.	.	.	.	.	.

*Proof.* Let  $s_h$  ( $1 \leq h \leq k$ ) be the sequences of unary tuples that make  $S_k$ . Let  $S_h$  be the partial sums from level  $h = 1$  to level  $h = k$ . By definition, we have  $S_h = S_{h-1} + s_h$ . Let  $m_{h-1}$  and  $m_h$  be the period of  $S_{h-1}$  and  $S_h$  respectively. Let  $n$  be the index of a given *permitted*  $(h-1)$ -tuple within the period of the partial sum  $S_{h-1}$ . Let us denote by  $r$  ( $1 \leq r < p_h$ ) one of the not *selected* remainders within the sequence  $s_h$ . We have the following system of 2 simultaneous congruences:

$$\begin{aligned} n' &\equiv n \pmod{m_{h-1}} \\ n' &\equiv r \pmod{p_h} \end{aligned}$$

Because  $(m_{h-1}, p_h) = 1$ , from the Chinese Remainder Theorem [5] there exists only one integer  $n'$  modulo  $m_{h-1}p_h = m_h$  solving the system. In other words, for a given *permitted*  $(h-1)$ -tuple at position  $n$  within the period of the partial sum  $S_{h-1}$  and a given not *selected* remainder  $r$  within the period  $p_h$  of the sequence  $s_h$ , we get one  $h$ -tuple at position  $n'$ , within the period of the partial sum  $S_h$ , by juxtaposition of the given *permitted*  $(h-1)$ -tuple at position  $n$  with the remainder  $r$ . Since the  $(h-1)$ -tuple at position  $n$  within the period of the partial sum  $S_{h-1}$  is a *permitted*  $(h-1)$ -tuple, and  $r$  is not a *selected* remainder, the  $h$ -tuple at position  $n'$  within the partial sum  $S_h$  must be a *permitted*  $h$ -tuple. Let  $c_h$  denote the number of *permitted*  $h$ -tuples within a period of the partial sum  $S_h$ . By definition, the number of not *selected* remainders within a period of the sequences of unary tuples  $s_h$  ( $1 < h \leq k$ ) is equal to  $p_h - 2$ . Therefore, the number of *permitted*  $h$ -tuples within a period of the partial sum  $S_h$  must be  $c_h = c_{h-1}(p_h - 2)$ . On the other hand, for level  $h = 1$ , within the period  $m_1 = p_1 = 2$  of the sequence of unary tuples  $s_1$  we have only one *selected* remainder. Thus, the number of *permitted* 1-tuples within a period of  $s_1$  is equal to  $c_1 = p_1 - 1$ . Consequently, the number of *permitted*  $k$ -tuples within a period of the partial sum  $S_k$  is given recursively by the formula:

$$\begin{aligned} c_1 &= (p_1 - 1) \\ c_h &= c_{h-1}(p_h - 2) \end{aligned}$$

It follows that  $c_k = (p_1 - 1) (p_2 - 2) (p_3 - 2) \dots (p_k - 2)$ . ■

**Lemma 2.3.** Let  $S_k$  be a given partial sum. Let  $m_k$  be the period of the partial sum  $S_k$  and let  $c_k$  be the number of *permitted*  $k$ -tuples within the period of  $S_k$ . We have  $c_k = o(m_k)$ .

Table 2.3: Partial sum  $S_4$  with *selected* remainders

$n$	2	3	5	7
1	1	1	1	1
2	[0]	[2]	2	2
3	1	[0]	[3]	[3]
4	[0]	1	4	4
5	1	[2]	[0]	[5]
6	[0]	[0]	1	6
7	1	1	2	0
8	[0]	[2]	[3]	1
9	1	[0]	4	2
10	[0]	1	[0]	[3]
11	1	[2]	1	4
12	[0]	[0]	2	[5]
13	1	1	[3]	6
14	[0]	[2]	4	0
15	1	[0]	[0]	1
16	[0]	1	1	2
.	.	.	.	.

*Proof.* For a given level  $k$ , from Proposition 2.1 and Proposition 2.2 we have:

$$\begin{aligned} \frac{c_k}{m_k} &= \frac{(p_1 - 1)(p_2 - 2)(p_3 - 2) \dots (p_k - 2)}{p_1 p_2 p_3 \dots p_k} = \frac{p_1 - 1}{p_1} \frac{p_2 - 2}{p_2} \frac{p_3 - 2}{p_3} \dots \frac{p_k - 2}{p_k} = \\ &= \left(1 - \frac{1}{p_1}\right) \left(1 - \frac{2}{p_2}\right) \left(1 - \frac{2}{p_3}\right) \dots \left(1 - \frac{2}{p_k}\right) \end{aligned}$$

Therefore  $c_k/m_k$  is a partial product of the infinite product:

$$\left(1 - \frac{1}{2}\right) \prod_{k=2}^{\infty} \left(1 - \frac{2}{p_k}\right)$$

which diverges to 0 if the series:

$$\frac{1}{2} + \sum_{k=2}^{\infty} \frac{2}{p_k} = \frac{1}{2} + 2 \sum_{k=2}^{\infty} \frac{1}{p_k} = \frac{1}{2} + 2 \left( \sum_{k=1}^{\infty} \frac{1}{p_k} - \frac{1}{2} \right) \quad (2.1)$$

diverges (see [4], Pag. 636). Indeed, the series (2.1) diverges, because it includes the reciprocal prime series, which diverges (see [5], Pag. 22). Thus we have:

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{1}{p_k} \rightarrow \infty &\implies \frac{1}{2} + \sum_{k=2}^{\infty} \frac{2}{p_k} \rightarrow \infty \implies \left(1 - \frac{1}{2}\right) \prod_{k=2}^{\infty} \left(1 - \frac{2}{p_k}\right) \rightarrow 0 \implies \\ &\implies \lim_{k \rightarrow \infty} \frac{c_k}{m_k} = 0 \end{aligned}$$

as desired. ■

**Lemma 2.4.** *Let  $S_k$  be a given partial sum. Let  $p_k$  be the characteristic prime of the partial sum  $S_k$ . Let  $c_k$  be the number of permitted  $k$ -tuples within the period of  $S_k$ . We have  $p_k^2 = o(c_k)$ .*

*Proof.* We have:



$$\begin{aligned} \frac{p_k^2}{c_k} &= \frac{p_k^2}{(p_1 - 1)(p_2 - 2)(p_3 - 2) \dots (p_k - 2)} = \\ &= \frac{1}{(p_1 - 1)(p_2 - 2)(p_3 - 2) \dots (p_{k-2} - 2)} \frac{p_k}{p_{k-1} - 2} \frac{p_k}{p_k - 2} \end{aligned} \quad (2.2)$$

Let  $g_k$  denote the gap  $p_{k+1} - p_k$ . Thus we have  $p_k / (p_{k-1} - 2) = (p_{k-1} + g_{k-1}) / (p_{k-1} - 2)$ . From the Bertrand-Chebyshev theorem we have  $g_{k-1} < p_{k-1} \implies (p_{k-1} + g_{k-1}) / (p_{k-1} - 2) < (2p_{k-1}) / (p_{k-1} - 2)$ . Therefore we have  $\lim_{k \rightarrow \infty} p_k / (p_{k-1} - 2) < 2$ , and consequently, returning to Eq. (2.2), we have  $\lim_{k \rightarrow \infty} p_k^2 / c_k = 0$ . ■

**Corollary 2.5.** *From Lemma 2.3 and Lemma 2.4 we have  $p_k^2 = o(c_k) \wedge c_k = o(m_k) \implies p_k^2 = o(m_k)$*

**Definition 2.11.** Let  $S_k$  be a given partial sum of the series  $\{S_k\}$ . The  $k$ -tuples of the sequence  $S_k$  have an order relation given by the index  $n$ . We define an interval of  $k$ -tuples, denoted by  $\mathbb{I}[x, y]$ , to be the set of consecutive  $k$ -tuples associated with an interval  $[x, y]$ , where  $x$  is the index of the first  $k$ -tuple and  $y$  is the index of the last  $k$ -tuple. Its size is the number of  $k$ -tuples in the interval and is equal to  $y - x + 1$ .

**Proposition 2.6.** *Construction Procedure*

Let  $S_k$  and  $S_{k+1}$  be partial sums of the series  $\sum s_k$ . Let  $m_k$  and  $m_{k+1}$  be the periods of  $S_k$  and  $S_{k+1}$  respectively. We have the following procedure: First we take  $p_{k+1}$  periods of the partial sum  $S_k$ . Next we juxtapose the remainders of the sequence  $s_{k+1}$  to the right of each  $k$ -tuple of  $S_k$  ( $S_k + s_{k+1}$ ). We get a whole period of the partial sum  $S_{k+1}$ .

*Proof.* From Proposition 2.1 the period  $m_k$  of the partial sum  $S_k$  is equal to  $m_k = p_1 \cdot p_2 \cdot p_3 \cdots p_k$ . If we repeat  $p_{k+1}$  times the period of the partial sum  $S_k$  the total number of  $k$ -tuples will be  $m_k \cdot p_{k+1} = p_1 \cdot p_2 \cdot p_3 \cdots p_k \cdot p_{k+1} = m_{k+1}$ . From these  $k$ -tuples of  $S_k$ , when we sum the sequence of unary tuples  $s_{k+1}$ , the number of  $(k+1)$ -tuples of  $S_{k+1}$  that we obtain is equal to  $m_{k+1}$  either, that is to say, a period of  $S_{k+1}$ . ■

## Chapter 3

# The density of *permitted* $k$ -tuples

**Definition 3.1.** Let  $S_k$  be a partial sum of the series  $\sum s_k$ . Let  $\mathbb{I}[x, y]$  be an interval of  $k$ -tuples of size  $y - x + 1$ . Let  $(y - x + 1)/p_k$  be the number of subintervals of size  $p_k$ . Let  $c$  denote the number of *permitted*  $k$ -tuples within  $\mathbb{I}[x, y]$ . We define the density of *permitted*  $k$ -tuples in the interval  $\mathbb{I}[x, y]$  by the equation:

$$\delta = \frac{c}{(y - x + 1)/p_k}$$

**Example 3.1.** The size of the period  $\mathbb{I}[1, 210]$  of the partial sum  $S_k = S_4$  is equal to  $m = 2 \times 3 \times 5 \times 7 = 30 \times 7 = 210$  (30 intervals of size 7), and the number of *permitted* 4-tuples within the period is equal to  $c = (2 - 1)(3 - 2)(5 - 2)(7 - 2) = 15$ . Consequently, the density of *permitted* 4-tuples is:

$$\delta = \frac{c}{30} = \frac{15}{30} = 0.5$$

**Remark 3.1.** The density of *permitted*  $k$ -tuples represents the average number of *permitted*  $k$ -tuples inside subintervals of size  $p_k$ .

**Definition 3.2.** We use the notation  $\delta_k$  to denote the density of *permitted*  $k$ -tuples within a period of the partial sum  $S_k$ .

**Proposition 3.1.** *We have:*

$$\delta_k = \frac{p_1 - 1}{p_1} \frac{p_2 - 2}{p_2} \frac{p_3 - 2}{p_3} \dots \frac{p_{k-1} - 2}{p_{k-1}} (p_k - 2)$$

*Proof.* Within a period of the partial sum  $S_k$ , the total number of  $k$ -tuples is equal to  $p_1 \cdot p_2 \cdot p_3 \cdot p_4 \dots p_{k-1} \cdot p_k$  (Proposition 2.1). Therefore the number of intervals of size  $p_k$  is equal to:

$$\frac{p_1 \cdot p_2 \cdot p_3 \cdot p_4 \dots p_{k-1} \cdot p_k}{p_k} = p_1 \cdot p_2 \cdot p_3 \cdot p_4 \dots p_{k-1}$$

On the other hand, the number of *permitted*  $k$ -tuples within a period of  $S_k$  is equal to  $(p_1 - 1)(p_2 - 2)(p_3 - 2) \dots (p_{k-1} - 2)(p_k - 2)$  (Proposition 2.2). Consequently, the density  $\delta_k$  of *permitted*  $k$ -tuples in a period of  $S_k$  is equal to:

$$\begin{aligned} \delta_k &= \frac{(p_1 - 1)(p_2 - 2)(p_3 - 2) \dots (p_{k-1} - 2)(p_k - 2)}{p_1 \cdot p_2 \cdot p_3 \cdot p_4 \dots p_{k-1}} = \\ &= \frac{p_1 - 1}{p_1} \frac{p_2 - 2}{p_2} \frac{p_3 - 2}{p_3} \dots \frac{p_{k-1} - 2}{p_{k-1}} (p_k - 2) \end{aligned}$$

■

**Proposition 3.2.** *Let  $S_k$  and  $S_{k+1}$  be partial sums of the series  $\sum s_k$ . If  $\delta_k$  denotes the density of *permitted*  $k$ -tuples within a period of  $S_k$  and  $\delta_{k+1}$  denotes the density of *permitted*  $(k+1)$ -tuples within a period of  $S_{k+1}$  we have:*

$$\delta_{k+1} = \delta_k \frac{(p_{k+1} - 2)}{p_k}$$

*Proof.* From Proposition 3.1 the density  $\delta_k$  of *permitted*  $k$ -tuples within a period of  $S_k$  is equal to:

$$\delta_k = \frac{p_1 - 1}{p_1} \frac{p_2 - 2}{p_2} \frac{p_3 - 2}{p_3} \dots \frac{p_{k-1} - 2}{p_{k-1}} (p_k - 2)$$

and within a period of the partial sum  $S_{k+1}$  the density  $\delta_{k+1}$  of *permitted*  $(k+1)$ -tuples is equal to:

$$\delta_{k+1} = \frac{p_1 - 1}{p_1} \frac{p_2 - 2}{p_2} \frac{p_3 - 2}{p_3} \dots \frac{p_{k-1} - 2}{p_{k-1}} \frac{p_k - 2}{p_k} (p_{k+1} - 2)$$

Consequently, making the quotient and simplifying expression we have:

$$\frac{\delta_{k+1}}{\delta_k} = \frac{(p_{k+1} - 2)}{p_k} \implies \delta_{k+1} = \delta_k \frac{(p_{k+1} - 2)}{p_k}$$

■

**Corollary 3.3.** *From Proposition 3.2 we have:*

1.  $p_{k+1} - p_k < 2 \implies \delta_{k+1} < \delta_k$ .
2.  $p_{k+1} - p_k = 2 \implies \delta_{k+1} = \delta_k$ .
3.  $p_{k+1} - p_k > 2 \implies \delta_{k+1} > \delta_k$ .

**Example 3.2.** The characteristic primes of the partial sums  $S_4$  and  $S_5$  are  $p_4 = 7$  and  $p_5 = 11$ . The period  $m_4$  of the partial sum  $S_4$  is equal to  $m_4 = 2 \times 3 \times 5 \times 7 = 30 \times 7 = 210$  and the number of *permitted* 4-tuples is equal to  $c_4 = (2 - 1) (3 - 2) (5 - 2) (7 - 2) = 15$ . Then, within a period of  $S_4$  the density of *permitted* 4-tuples is equal to:

$$\delta_4 = \frac{15}{30} = 0.500$$

The period  $m_5$  of the partial sum  $S_5$  is equal to  $m_5 = 2 \times 3 \times 5 \times 7 \times 11 = 210 \times 11 = 2310$  and the number of *permitted* 5-tuples is equal to  $c_5 = (2 - 1) (3 - 2) (5 - 2) (7 - 2) (11 - 2) = 135$ . Then, within a period of  $S_5$  the density of *permitted* 5-tuples is:

$$\delta_5 = \frac{135}{210} \approx 0.643$$

Note that since 7 and 11 are not twin primes we have  $\delta_5 > \delta_4$  (see Corollary 3.3).

Proposition 3.2 asserts that the density  $\delta_k$  grows up. Now we will prove that  $\delta_k \rightarrow \infty$ .

**Definition 3.3.** Let  $p_h > 2$  and  $p_{h+1}$  be consecutive primes. We denote by  $\Delta_h$  the difference  $p_{h+1} - p_h - 2$ .

**Theorem 3.4.** *Let  $S_k$  be a given partial sum. Let  $m_k$  be the period of the partial sum  $S_k$  and let  $c_k$  be the number of permitted  $k$ -tuples within the period of  $S_k$ . Let  $\delta_k$  be the density of permitted  $k$ -tuples within a period of  $S_k$ . We have  $\delta_k \rightarrow \infty$ .*

*Proof.* From Proposition 3.1 we have:

$$\delta_k = \frac{p_1 - 1}{p_1} \frac{p_2 - 2}{p_2} \frac{p_3 - 2}{p_3} \frac{p_4 - 2}{p_4} \frac{p_5 - 2}{p_5} \dots \frac{p_{k-1} - 2}{p_{k-1}} (p_k - 2)$$

If we shift denominators to the right we have:

$$\delta_k = (p_1 - 1) \frac{p_2 - 2}{p_1} \frac{p_3 - 2}{p_2} \frac{p_4 - 2}{p_3} \frac{p_5 - 2}{p_4} \dots \frac{p_{k-1} - 2}{p_{k-2}} \frac{p_k - 2}{p_{k-1}}$$

By definition,  $\Delta_k = p_{k+1} - p_k - 2 \implies p_{k+1} - 2 = p_k + \Delta_k$ . Consequently, we can write the expression of  $\delta_k$  this way:

$$\begin{aligned}
\delta_k &= \frac{1}{2} \frac{p_2 + \Delta_2}{p_2} \frac{p_3 + \Delta_3}{p_3} \frac{p_4 + \Delta_4}{p_4} \dots \frac{p_{k-2} + \Delta_{k-2}}{p_{k-2}} \frac{p_{k-1} + \Delta_{k-1}}{p_{k-1}} = \\
&= \frac{1}{2} \left(1 + \frac{\Delta_2}{p_2}\right) \left(1 + \frac{\Delta_3}{p_3}\right) \left(1 + \frac{\Delta_4}{p_4}\right) \dots \left(1 + \frac{\Delta_{k-2}}{p_{k-2}}\right) \left(1 + \frac{\Delta_{k-1}}{p_{k-1}}\right) = \\
&= \frac{1}{3} \left[ \left(1 + \frac{1}{p_1}\right) \left(1 + \frac{\Delta_2}{p_2}\right) \left(1 + \frac{\Delta_3}{p_3}\right) \left(1 + \frac{\Delta_4}{p_4}\right) \dots \left(1 + \frac{\Delta_{k-1}}{p_{k-1}}\right) \left(1 + \frac{\Delta_k}{p_k}\right) \right] \frac{p_k}{p_k + \Delta_k}
\end{aligned} \tag{3.1}$$

The expression between square brackets is a partial product of the infinite product:

$$\left(1 + \frac{1}{p_1}\right) \prod_{k=2}^{\infty} \left(1 + \frac{\Delta_k}{p_k}\right) \tag{3.2}$$

which diverges if the series:

$$\frac{1}{p_1} + \sum_{k=2}^{\infty} \frac{\Delta_k}{p_k} \tag{3.3}$$

diverges (see [4], Pag. 636). In the series (3.3), by definition, if  $p_k$  is the first of a pair of twin primes we have  $\Delta_k = 0$  and otherwise we have  $\Delta_k > 0$ . Let  $\sum_{j=1}^{\infty} 1/q_j$  denote the series where every prime  $q_j$  is the first of a pair of twin primes. Since the series of the reciprocals of all the twin primes converges [10], the series  $\sum_{j=1}^{\infty} 1/q_j$  converges either. Therefore the series  $\sum_{k=1}^{\infty} 1/p_k - \sum_{j=1}^{\infty} 1/q_j$  diverges, because  $\sum_{k=1}^{\infty} 1/p_k$  diverges (see [5], Pag. 22). By comparison to the series  $\sum_{k=1}^{\infty} 1/p_k - \sum_{j=1}^{\infty} 1/q_j$  we have that the series (3.3) also diverges, because  $\Delta_k/p_k > 1/p_k$  for the terms where  $\Delta_k > 0$ . Thus we have that the infinite product (3.2) tends to  $\infty$  either. On the other hand, from the Bertrand-Chebyshev theorem we have  $\Delta_k < p_k \implies p_k/(p_k + \Delta_k) > 1/2$ . Consequently, as  $k \rightarrow \infty$ , the last expression in (3.1) is dominated by (3.2) and thus we have  $\delta_k \rightarrow \infty$ , as desired. ■

## Chapter 4

# The average number of *permitted* $k$ -tuples within a given interval $\mathbb{I}[x, y]$

In this chapter we will calculate the average number of *permitted*  $k$ -tuples within a given interval  $\mathbb{I}[x, y]$  of the partial sum  $S_k$ .

**Definition 4.1.** Let  $s_h$  ( $1 \leq h \leq k$ ) be the sequences of unary tuples that make the partial sum  $S_k$ . A given choice of the *selected* remainders within the period of one of the sequences  $s_h$  or within the periods of all the sequences  $s_h$  ( $1 \leq h \leq k$ ) will be called a *combination* of *selected* remainders. We often denote by  $\nu$  the number of *combinations* of *selected* remainders within the period of one sequence  $s_h$ . Since, by definition, within the sequences  $s_h$  ( $1 < h \leq k$ ) we have two *selected* remainders within the period  $p_h$ , we have:

$$\nu = \binom{p_h}{2}$$

Within the sequence  $s_1$  we have only one *selected* remainder within the period  $p_1 = 2 \implies \nu = 2$ . We denote by  $\nu_k$  the number of *combinations* of *selected* remainders within the periods of all of the sequences  $s_h$  ( $1 \leq h \leq k$ ). Therefore we have:

$$\nu_k = \binom{p_1}{1} \binom{p_2}{2} \binom{p_3}{2} \cdots \binom{p_k}{2}$$

**Definition 4.2.** Type I operation.

Let  $s_h$  ( $1 \leq h \leq k$ ) be the sequences of unary tuples that make the partial sum  $S_k$ . For  $h > 1$  let  $r, r' \pmod{p_h}$  be the *selected* remainders within a period  $p_h$  of the sequence  $s_h$ . We define the operation of changing the *selected* remainders  $r, r' \pmod{p_h}$  by  $r + 1, r' + 1 \pmod{p_h}$  to be Type I operation.

For the sequence  $s_1$ , we also define the operation of changing the *selected* remainder  $r \pmod{p_1}$  by  $r + 1 \pmod{p_1}$  to be Type I operation.

**Definition 4.3.** Type II operation.

Let  $s_h$  ( $1 < h \leq k$ ) be the sequences of unary tuples that make the partial sum  $S_k$ . Let  $r, r' \pmod{p_h}$  be the *selected* remainders within a period  $p_h$  of the sequence  $s_h$ . We define to be Type II operation the following operation:

- 1) One of the *selected* remainders will not be changed. Let  $r$  be the *selected* remainder that will not be changed.
- 2) Next we change the other *selected* remainder  $r' \pmod{p_h}$  by  $r' + 1 \pmod{p_h}$ ,  $r \neq r' + 1$  or  $r' \pmod{p_h}$  by  $r' - 1 \pmod{p_h}$ ,  $r \neq r' - 1$ .

**Proposition 4.1.** Let  $S_k$  be a given partial sum of level  $k \geq 1$ . Let  $s_{k+1}$  be the sequence of unary tuples of level  $k + 1$ . Let  $r, r'$  ( $0 \leq r, r' < p_{k+1}$ ) be the *selected* remainders within the periods of  $s_{k+1}$ .

Assume that  $n \in \mathbb{N}$  denotes the index of a given *permitted*  $k$ -tuple of  $S_k$ . When we juxtapose the remainders of the sequence  $s_{k+1}$  to the right of each  $k$ -tuple of  $S_k$ , if  $n \equiv r \pmod{p_{k+1}} \vee n \equiv r' \pmod{p_{k+1}}$ , we get at position  $n$  a *prohibited*  $(k + 1)$ -tuple of  $S_{k+1}$ . If  $n \not\equiv r \pmod{p_{k+1}} \wedge n \not\equiv r' \pmod{p_{k+1}}$ , the *permitted*  $k$ -tuple at position  $n$  remains as *permitted*  $(k + 1)$ -tuple of  $S_{k+1}$ .

*Proof.* By definition we have  $S_{k+1} = S_k + s_{k+1}$ , that is to say, we get the partial sum  $S_{k+1}$  by juxtaposing each element of the sequence  $s_{k+1}$  to the right of each  $k$ -tuple of  $S_k$ . The juxtaposition of one of the two *selected* remainders of  $s_{k+1}$  to the right of any  $k$ -tuple of  $S_k$  always results, by definition, in a *prohibited*  $(k + 1)$ -tuple of  $S_{k+1}$ . If the juxtaposed remainder is not one of the two *selected* remainders of  $s_{k+1}$ , the *permitted*  $k$ -tuple at position  $n$  will be a *permitted*  $(k + 1)$ -tuple of  $S_{k+1}$ . ■

**Definition 4.4.** Let  $S_k$  and  $S_{k+1}$  be the partial sums of level  $k$  and  $k+1$  respectively ( $k \geq 1$ ). Let  $s_{k+1}$  be the sequence of unary tuples of level  $k+1$ . Let  $\mathbb{I}[x, y]_k$  be an interval of  $k$ -tuples within  $S_k$ , and let  $\mathbb{I}[x, y]_{k+1}$  be an interval of  $(k+1)$ -tuples within  $S_{k+1}$ . When we juxtapose the remainders of the sequence  $s_{k+1}$  to the right of each  $k$ -tuple of  $S_k$ , from Proposition 4.1, the *permitted*  $k$ -tuples of  $S_k$  that are congruent with a given *selected* remainder of  $s_{k+1}$  are converted to a *prohibited*  $(k+1)$ -tuples of  $S_{k+1}$ . We denote by  $f$  the fraction of the *permitted*  $k$ -tuples within the interval  $\mathbb{I}[x, y]_k$  that are converted to *prohibited*  $(k+1)$ -tuples within the interval  $\mathbb{I}[x, y]_{k+1}$ . For the partial sum  $S_1$  ( $k=1$ ),  $f$  denotes the fraction of the *prohibited* 1-tuples within the interval  $\mathbb{I}[x, y]_{k=1}$ .

We denote by  $\bar{f}$  the average of  $f$  for all the *combinations* of *selected* remainders within the sequence  $s_{k+1}$  ( $k \geq 1$ ). For the partial sum  $S_1$  ( $k=1$ ),  $\bar{f}$  denotes the average of  $f$  for the 2 *combinations* of *selected* remainders within the sequence  $s_1$ .

**Proposition 4.2.** For  $k \geq 1$  we have  $\bar{f} = 2/p_{k+1}$ . For  $S_1$  we have  $\bar{f} = 1/p_1$ .

*Proof.* Let  $[0], [1], [2], \dots, [p_{k+1}-1]$  be the residue classes  $\pmod{p_{k+1}}$ . Let  $c$  be the number of *permitted*  $k$ -tuples within  $\mathbb{I}[x, y]_k$ . We denote by  $c_0, c_1, c_2, \dots, c_{p_{k+1}-1}$  the number of *permitted*  $k$ -tuples in the residue classes  $[0], [1], [2], \dots, [p_{k+1}-1]$ . Therefore we have  $c = c_0 + c_1 + c_2 + \dots + c_{p_{k+1}-1}$ . Let  $r, r'$  ( $0 \leq r, r' < p_{k+1}$ ,  $r \neq r'$ ) be the *selected* remainders within the periods of  $s_{k+1}$ . The number of ways of choosing 2 out of  $p_{k+1}$  *selected* remainders is equal to:

$$\nu = \binom{p_{k+1}}{2} = \frac{p_{k+1}!}{2! (p_{k+1}-2)!} = \frac{(p_{k+1}-1)}{2} p_{k+1}$$

If we denote by  $\nu_I$  the number of *combinations* of *selected* remainders that we can reach by repeated Type I operations, we have  $\nu_I = p_{k+1}$ . If we fix  $r$  as one of the *selected* remainders, denoting by  $\nu_{II}$  the number of *combinations* of *selected* remainders that we can reach by repeated Type II operations, we have  $\nu_{II} = (p_{k+1}-1)/2$ .

If  $r, r'$  are the *selected* remainders, the fraction of the  $c$  *permitted*  $k$ -tuples within the interval  $\mathbb{I}[x, y]_k$  of  $S_k$  that are converted to *prohibited*  $(k+1)$ -tuples within the interval  $\mathbb{I}[x, y]_{k+1}$  of  $S_{k+1}$  is equal to  $(c_r + c_{r'})/c$ . Averaging for the  $\nu_I$  *combinations* of *selected* remainders that we can reach by repeated Type I operations we have:

$$\frac{\sum_{i=1}^{\nu_I} \frac{c_r + c_{r'}}{c}}{\nu_I} = \frac{\sum_{i=1}^{p_{k+1}} \frac{c_r + c_{r'}}{c}}{p_{k+1}} = \frac{\frac{1}{c} \sum_{r=0}^{p_{k+1}-1} c_r + \frac{1}{c} \sum_{r'=0}^{p_{k+1}-1} c_{r'}}{p_{k+1}} = \frac{2}{p_{k+1}}$$

Now, if we average for the  $\nu_{II}$  *combinations* of *selected* remainders that we can reach by repeated Type II operations we have:

$$\bar{f} = \frac{\sum_{j=1}^{\nu_{II}} \frac{2}{p_{k+1}}}{\nu_{II}} = \frac{2}{p_{k+1}}$$

For the partial sum  $S_1$  ( $k=1$ ), we have two residue classes  $\pmod{p_1=2}$  and 1 *selected* remainder. Therefore  $\bar{f} = 1/p_1$ . ■

**Definition 4.5.** When we juxtapose the remainders of the sequence  $s_{k+1}$  to the right of each  $k$ -tuple of  $S_k$ , from Proposition 4.1, the *permitted*  $k$ -tuples of  $S_k$  that are not congruents with any of the two *selected* remainders of  $s_{k+1}$  are kept as *permitted*  $(k+1)$ -tuples of  $S_{k+1}$ . We denote by  $f'$  the fraction of *permitted*  $k$ -tuples within the interval  $\mathbb{I}[x, y]_k$  of  $S_k$ , that are transferred to the interval  $\mathbb{I}[x, y]_{k+1}$  of  $S_{k+1}$  as *permitted*  $(k+1)$ -tuples. For the partial sum  $S_1$  ( $k=1$ ),  $f'$  denotes the fraction of the *permitted* 1-tuples within the interval  $\mathbb{I}[x, y]_{k=1}$ .

We denote by  $\bar{f}'$  the average of  $f'$  for all the *combinations* of *selected* remainders within the sequence  $s_{k+1}$  ( $k \geq 1$ ). For the partial sum  $S_1$  ( $k=1$ ),  $\bar{f}'$  denotes the average of  $f'$  for the 2 *combinations* of *selected* remainders within the sequence  $s_1$ .

**Proposition 4.3.** For  $k \geq 1$  we have  $\bar{f}' = (p_{k+1}-2)/p_{k+1}$ . For  $S_1$  we have  $\bar{f}' = (p_1-1)/p_1$ .

*Proof.* Using Proposition 4.2, we have:

$$\bar{f}' = 1 - \bar{f} = 1 - \frac{2}{p_{k+1}} = \frac{p_{k+1}-2}{p_{k+1}}$$

For the partial sum  $S_1$  ( $k=1$ ), we have  $\bar{f} = 1/p_1 \implies \bar{f}' = (p_1-1)/p_1$ . ■

**Definition 4.6.** Let  $S_k$  be the partial sum of level  $k$ . Let  $s_h$  ( $1 \leq h \leq k$ ) be the sequences of unary tuples that make  $S_k$ . Let  $\mathbb{I}[x, y]_k$  be an interval of  $k$ -tuples within  $S_k$ . We denote by  $\bar{c}_k^{\mathbb{I}}$  the average number of *permitted*  $k$ -tuples within the interval  $\mathbb{I}[x, y]_k$ , for all the *combinations* of *selected* remainders within the periods of the sequences  $s_h$ . We denote by  $\bar{\delta}_k^{\mathbb{I}}$  the average density of *permitted*  $k$ -tuples within the interval  $\mathbb{I}[x, y]_k$ , for all the *combinations* of *selected* remainders within the periods of the sequences  $s_h$ .

**Theorem 4.4.** Let  $\delta_k$  be the density of *permitted*  $k$ -tuples within a period of the partial sum  $S_k$ . We have  $\bar{\delta}_k^{\mathbb{I}} = \delta_k$ .

*Proof.* If there were not *selected* remainders in the sequences  $s_h$  ( $1 \leq h \leq k$ ), all the  $k$ -tuples within the interval  $\mathbb{I}[x, y]_k$  would be *permitted*  $k$ -tuples  $\implies \bar{c}_k^{\mathbb{I}} = (y - x + 1)$ , where  $(y - x + 1)$  represents the size of the interval  $\mathbb{I}[x, y]_k$ . But since we have *selected* remainders in the sequences  $s_h$ , using Proposition 4.3 at each transition level from  $h = 1$  to  $h = k$ , we have:

$$\bar{c}_k^{\mathbb{I}} = (y - x + 1) \frac{(p_1 - 1)}{p_1} \frac{(p_2 - 2)}{p_2} \frac{(p_3 - 2)}{p_3} \dots \frac{(p_k - 2)}{p_k}$$

The number of intervals of size  $p_k$  within the interval  $\mathbb{I}[x, y]_k$  is equal to  $(y - x + 1) / p_k$ . Therefore, from Proposition 3.1 we have:

$$\begin{aligned} \bar{\delta}_k^{\mathbb{I}} &= \frac{\bar{c}_k^{\mathbb{I}}}{\frac{(y-x+1)}{p_k}} = \frac{p_k}{(y - x + 1)} \bar{c}_k^{\mathbb{I}} = \frac{p_k}{(y - x + 1)} \left( (y - x + 1) \frac{(p_1 - 1)}{p_1} \frac{(p_2 - 2)}{p_2} \frac{(p_3 - 2)}{p_3} \dots \frac{(p_k - 2)}{p_k} \right) = \\ &= \frac{(p_1 - 1)}{p_1} \frac{(p_2 - 2)}{p_2} \frac{(p_3 - 2)}{p_3} \dots (p_k - 2) = \delta_k \end{aligned}$$

■

## Chapter 5

# The density of *permitted* $k$ -tuples within the interval $\mathbb{I}[1, y]$ for $y \rightarrow \infty$

**Definition 5.1.** Let  $S_k$  be a given partial sum of level  $k$ . Let  $p_k$  be the characteristic prime and let  $m_k$  be the period of  $S_k$ . Let  $\delta_k$  be the density of *permitted*  $k$ -tuples within the period of  $S_k$ . Let  $\mathbb{I}[1, y]$  be a given interval of  $k$ -tuples within the partial sum  $S_k$ , where  $y \geq m_k$ . We denote by  $c_k^{\mathbb{I}}$  the number of *permitted*  $k$ -tuples and by  $\delta_k^{\mathbb{I}}$  the density of *permitted*  $k$ -tuples within  $\mathbb{I}[1, y]$ . We denote by  $\eta_k$  or alternatively by  $\eta$  the integer part:

$$\eta_k = \left\lfloor \frac{y}{m_k} \right\rfloor$$

and we denote by  $\epsilon_k$  or alternatively by  $\epsilon$  the residual part:

$$\epsilon_k = \left\{ \frac{y}{m_k} \right\}$$

We denote by  $c_\eta$  the number of *permitted*  $k$ -tuples within  $\mathbb{I}[1, \eta_k m_k] \subseteq \mathbb{I}[1, y]$  and we denote by  $c_\epsilon$  the number of *permitted*  $k$ -tuples within  $\mathbb{I}[\eta_k m_k + 1, y] \subseteq \mathbb{I}[1, y]$ .

**Proposition 5.1.** *We have:*

$$\delta_k^{\mathbb{I}} = \frac{\eta_k m_k}{y} \delta_k + \frac{p_k c_\epsilon}{y}$$

*Proof.* Let  $c_k^{\mathbb{I}}$  be the number of *permitted*  $k$ -tuples within  $\mathbb{I}[1, y]$ . By definition we have:

$$\delta_k^{\mathbb{I}} = \frac{c_k^{\mathbb{I}}}{\frac{y}{p_k}}$$

Since  $\eta_k$  represents the times that the period of  $S_k$  fits in the interval  $\mathbb{I}[1, y]$ , we have that  $\eta_k m_k$  is the part of the interval  $\mathbb{I}[1, y]$  that is multiple of  $m_k$ , and  $\eta_k m_k / p_k$  is the number of intervals of size  $p_k$  within this part of the interval  $\mathbb{I}[1, y]$ . Multiplying by the density of *permitted*  $k$ -tuples within the period of  $S_k$ , we have:

$$c_\eta = \frac{\eta_k m_k}{p_k} \delta_k$$

Therefore, we have:

$$\delta_k^{\mathbb{I}} = \frac{c_k^{\mathbb{I}}}{\frac{y}{p_k}} = \frac{c_\eta + c_\epsilon}{\frac{y}{p_k}} = \frac{\frac{\eta_k m_k}{p_k} \delta_k + c_\epsilon}{\frac{y}{p_k}} = \frac{\eta_k m_k}{y} \delta_k + \frac{p_k c_\epsilon}{y}$$

■

**Proposition 5.2.** *We have:*

$$\lim_{y \rightarrow \infty} \delta_k^{\mathbb{I}} = \delta_k$$



*Proof.* From Proposition 5.1 we have:

$$\delta_k^{\mathbb{I}} = \frac{\eta_k m_k}{y} \delta_k + \frac{p_k c_\epsilon}{y} = \left( \frac{\eta_k m_k}{y} \right) \delta_k + \frac{p_k c_\epsilon}{y}$$

If we take the limit for  $y \rightarrow \infty$  we have:

$$\lim_{y \rightarrow \infty} \delta_k^{\mathbb{I}} = \lim_{y \rightarrow \infty} \left( \frac{\eta_k m_k}{y} \right) \delta_k + \lim_{y \rightarrow \infty} \frac{p_k c_\epsilon}{y}$$

On the one hand we have:

$$\lim_{y \rightarrow \infty} \frac{\eta_k m_k}{y} = \lim_{y \rightarrow \infty} \frac{\left\lfloor \frac{y}{m_k} \right\rfloor m_k}{y} = 1$$

On the other hand, the number  $c_\epsilon$  of *permitted*  $k$ -tuples within  $\mathbb{I}[\eta_k m_k + 1, y] \subseteq \mathbb{I}[1, y]$  is less than the number of *permitted*  $k$ -tuples within the period of  $S_k$ . Therefore,  $c_\epsilon$  is upper bounded for a given level  $k$ . Besides, for a given level  $k$  the values  $p_k$  and  $\delta_k$  are constants. Therefore we have:

$$\lim_{y \rightarrow \infty} \delta_k^{\mathbb{I}} = \lim_{y \rightarrow \infty} \left( \frac{\eta_k m_k}{y} \right) \delta_k + \lim_{y \rightarrow \infty} \frac{p_k c_\epsilon}{y} = \delta_k + 0 = \delta_k$$

■

**Proposition 5.3.** *Let  $\mathbb{I}[1, y]$  be a given interval of  $k$ -tuples within the partial sum  $S_k$ , where  $y \geq m_k$ . We have:*

$$\frac{\eta_k m_k}{\eta_k m_k + (m_k - 1)} \delta_k \leq \delta_k^{\mathbb{I}} \leq \frac{(\eta_k + 1) m_k}{\eta_k m_k + 1} \delta_k$$

*Proof.* From Proposition 5.1 we have:

$$\delta_k^{\mathbb{I}} = \frac{\eta_k m_k}{y} \delta_k + \frac{p_k c_\epsilon}{y} \tag{5.1}$$

The number of intervals of size  $p_k$  within the period of  $S_k$  is equal to  $m_k/p_k$ . Consequently,  $\delta_k m_k/p_k$  represents the number of *permitted*  $k$ -tuples within the period of  $S_k$ . On the other hand, the number  $c_\epsilon$  of *permitted*  $k$ -tuples within  $\mathbb{I}[\eta_k m_k + 1, y] \subseteq \mathbb{I}[1, y]$  is less than the number of *permitted*  $k$ -tuples within the period of  $S_k$ . Therefore, we have  $0 \leq c_\epsilon < \delta_k m_k/p_k$ . If  $y$  is multiple of  $m_k$  we have  $y = \eta_k m_k \wedge c_\epsilon = 0 \implies \delta_k^{\mathbb{I}} = \delta_k$ . If  $y$  is not multiple of  $m_k$  we have  $\eta_k m_k + 1 \leq y \leq \eta_k m_k + (m_k - 1) \wedge 0 \leq c_\epsilon < \delta_k m_k/p_k$ . Consequently, replacing  $y$  in Eq. (5.1), we have:

$$\frac{\eta_k m_k}{\eta_k m_k + (m_k - 1)} \delta_k + \frac{p_k c_\epsilon}{\eta_k m_k + (m_k - 1)} \leq \delta_k^{\mathbb{I}} \leq \frac{\eta_k m_k}{\eta_k m_k + 1} \delta_k + \frac{p_k c_\epsilon}{\eta_k m_k + 1}$$

Finally, replacing  $c_\epsilon$  we have:

$$\frac{\eta_k m_k}{\eta_k m_k + (m_k - 1)} \delta_k \leq \delta_k^{\mathbb{I}} \leq \frac{\eta_k m_k}{\eta_k m_k + 1} \delta_k + \frac{\delta_k m_k}{\eta_k m_k + 1}$$

$$\frac{\eta_k m_k}{\eta_k m_k + (m_k - 1)} \delta_k \leq \delta_k^{\mathbb{I}} \leq \frac{(\eta_k + 1) m_k}{\eta_k m_k + 1} \delta_k$$

■

## Chapter 6

# The number of *permitted* $k$ -tuples within the interval $\mathbb{I} [1, p_k^2]$

To prove the Main Proposition we need to prove that for level  $k \geq 13$ , the number of *permitted*  $k$ -tuples within the interval  $\mathbb{I} [1, p_k^2]$  of the partial sum  $S_k$  is  $\geq 3$  and this is the aim of Chapter 6. We need the following definitions:

**Definition 6.1.** Let  $S_k$  and  $S_{k+j}$  be the partial sums of level  $k$  and  $k+j$ , ( $j > 0$ ) respectively. We use the notation  $p_k \rightarrow p_{k+j}$  or alternatively  $k \rightarrow k+j$  to denote the transition from level  $k$  to level  $k+j$ .

**Definition 6.2.** Let  $S_k$  and  $S_{k+1}$  be the partial sums of level  $k$  and  $k+1$  respectively. For the level transition  $p_k \rightarrow p_{k+1}$ , we call the difference  $p_{k+1} - p_k$  the *order* of the transition.

**Definition 6.3.** When we juxtapose the remainders of the sequence  $s_{k+1}$  to the right of each  $k$ -tuple of  $S_k$ , from Proposition 4.1, a given *permitted*  $k$ -tuple of  $S_k$  that is congruent with a *selected* remainder of  $s_{k+1}$  are converted to a *prohibited*  $(k+1)$ -tuple of  $S_{k+1}$ . In that case, we say that at the level transition  $k \rightarrow k+1$  one *permitted*  $k$ -tuple is *removed*.

**Definition 6.4.** Let  $S_k$  be a given partial sum. Let  $s_h$  ( $1 \leq h \leq k$ ) be the sequences of unary tuples that make the partial sum  $S_k$ . If we write the indice  $n$  of the sequences  $s_h$  from top to bottom, and the level  $k$  from left to right (see Table 2.1, Table 2.2 and Table 2.3) we say that the partial sum  $S_k$  is in Vertical Position. Now, suppose that the partial sum  $S_k$  is in Vertical Position and we rotate it 45 degrees counterclockwise. Thus the indice  $n$  of the sequences  $s_h$  grows from left to right, and the level  $k$  grows from the bottom up. In this case we say that the partial sum  $S_k$  is in Horizontal Position.

**Definition 6.5.** Let  $S_k$  be a given partial sum, in Horizontal Position. Let  $s_h$  ( $1 \leq h \leq k$ ) be the sequences of unary tuples that make the partial sum  $S_k$ . Let  $\nu_k$  be the number of *combinations* of *selected* remainders within the periods of the sequences  $s_h$  ( $1 \leq h \leq k$ ). Let  $m_k$  be the period of the partial sum  $S_k$ . Let  $\mathbb{I} [1, y]$  be a given interval of  $k$ -tuples within  $S_k$ , where  $y \in \mathbb{N}$ ,  $1 < y < m_k$ . Therefore, within the partial sum  $S_k$ , the interval  $\mathbb{I} [1, m_k]$  is divided into 2 intervals:  $\mathbb{I} [1, y]$ , that we call the Left interval and  $\mathbb{I} [y+1, m_k]$  that we call the Right interval.

Recall the notation  $c_k$  to denote the number of *permitted*  $k$ -tuples within the interval  $\mathbb{I} [1, m_k]$ . Recall the notation  $c_k^{\mathbb{I}}$  to denote the number of *permitted*  $k$ -tuples within the interval  $\mathbb{I} [1, y]$  and the notation  $\bar{c}_k^{\mathbb{I}}$  to denote the average number of *permitted*  $k$ -tuples within the interval  $\mathbb{I} [1, y]$ , for all the  $\nu_k$  *combinations* of *selected* remainders.

Now, we use the notation  $c_k^{\mathbb{L}}$  to denote the number of *permitted*  $k$ -tuples within the Left interval  $\mathbb{I} [1, y]$  and we use the notation  $c_k^{\mathbb{R}}$  to denote the number of *permitted*  $k$ -tuples within the Right interval  $\mathbb{I} [y+1, m_k]$ . Therefore we have  $c_k = c_k^{\mathbb{L}} + c_k^{\mathbb{R}}$ . The notation  $c_k^{\mathbb{L}}$  is an alternative notation for  $c_k^{\mathbb{I}}$ .

**Definition 6.6.** Let  $S_k$  be a given partial sum. Let  $s_h$  ( $1 \leq h \leq k$ ) be the sequences of unary tuples that make the partial sum  $S_k$ . Let  $\nu_k$  be the number of *combinations* of *selected* remainders within the periods of the sequences  $s_h$  ( $1 \leq h \leq k$ ).

Recall the notation  $\delta_k$  to denote the density of *permitted*  $k$ -tuples within the interval  $\mathbb{I} [1, m_k]$ . Recall the notation  $\delta_k^{\mathbb{I}}$  to denote the density of *permitted*  $k$ -tuples within  $\mathbb{I} [1, y]$  and the notation  $\bar{\delta}_k^{\mathbb{I}}$  to denote the average density of *permitted*  $k$ -tuples within the interval  $\mathbb{I} [1, y]$ , for all the  $\nu_k$  *combinations* of *selected* remainders within the sequences  $s_h$ . From Theorem 4.4 we have  $\bar{\delta}_k^{\mathbb{I}} = \delta_k$ . We often call  $\delta_k^{\mathbb{I}}$  the *true* density of *permitted*  $k$ -tuples to distinguish from the average density  $\delta_k$  within  $\mathbb{I} [1, y]$ .

Now, we use the notation  $\delta_k^{\mathbb{L}}$  to denote the *true* density of *permitted*  $k$ -tuples within the Left interval  $\mathbb{I} [1, y]$  and we use the notation  $\delta_k^{\mathbb{R}}$  to denote the *true* density of *permitted*  $k$ -tuples within the Right interval  $\mathbb{I} [y+1, m_k]$ . The notation  $\delta_k^{\mathbb{L}}$  is an alternative notation for  $\delta_k^{\mathbb{I}}$ .

**Definition 6.7.** We define the *true* increment  $\Delta(p_h \rightarrow p_{h+j})$ , ( $j > 0$ ) by the equation:  $\Delta(p_h \rightarrow p_{h+j}) = \delta_{h+j}^{\mathbb{I}} - \delta_h^{\mathbb{I}}$ . We define the average increment  $\overline{\Delta}(p_h \rightarrow p_{h+j})$  by the equation:  $\overline{\Delta}(p_h \rightarrow p_{h+j}) = \delta_{h+j} - \delta_h$ .

**Definition 6.8.** Let  $S_k$  be a given partial sum. Let  $c_k$  be the total number of *permitted*  $k$ -tuples within a period of  $S_k$ . Let  $i_j$  ( $j = 1, 2, 3, \dots, c_k$ ) be the indice of every *permitted*  $k$ -tuple within a period of  $S_k$ . We define the *positional pattern* of *permitted*  $k$ -tuples within a period of  $S_k$  to be the ordered set  $\{i_1, i_2, i_3, \dots, i_{c_k}\}$ .

**Definition 6.9.** Let  $S_k$  be a given partial sum. If  $i, j$  ( $j > i$ ) denotes the indices of two consecutive *permitted*  $k$ -tuples within  $S_k$ , we define the *distance* between the *permitted*  $k$ -tuples by the equation:

$$d_k(i, j) = |j - i|$$

**Proposition 6.1.** Let  $S_h$  be the partial sums from level  $h = 1$  to level  $h = k$  ( $k > 2$ ). Let  $s_h$  ( $1 \leq h \leq k$ ) be the sequences of unary tuples that make the partial sum  $S_k$ . Let  $c_h$  be the number of *permitted*  $h$ -tuples within a period of  $S_h$ . For any two consecutive *permitted*  $k$ -tuples within  $S_k$  we have  $d_k(i, j) = 6$  or  $d_k(i, j)$  multiple of 6.

*Proof.* By definition, within the periods of the sequences of unary tuples  $s_1$  and  $s_2$  there are 1 and 2 *selected* remainders, respectively. Therefore, if  $c_2$  is the number of *permitted* 2-tuples within the period of  $S_2$ , from Proposition 2.2 we have  $c_2 = (2 - 1)(3 - 2) = 1$ . Besides, from Proposition 2.1, the period for  $S_2$  is equal to  $m_2 = 2 \times 3 = 6$ . Consequently, at level  $h = 2$  we have  $d_2(i, j) = 6$ . From level  $h = 3$  to level  $h = k$  every sequence of unary tuples  $s_h$  removes *permitted*  $(h - 1)$ -tuples from the previous level. Then, for any two consecutive *permitted*  $k$ -tuples within the partial sum  $S_k$  the distance  $d_k(i, j)$  could be 6 or multiple of 6. ■

## 6.1 The true density $\delta_k^{\mathbb{I}}$ and the selected remainders

In this part we analyze how the number of *permitted*  $k$ -tuples within a given interval of the partial sum  $S_k$  depends on the *combination* of *selected* remainders within the periods of the sequences  $s_h$  ( $1 \leq h \leq k$ ).

Let us consider the number of *permitted*  $k$ -tuples within the intervals  $\mathbb{I}[1, m_k]$ ,  $\mathbb{I}[1, y]$  and  $\mathbb{I}[y + 1, m_k]$  within the partial sum  $S_k$ . From Proposition 2.2, the value of  $c_k$  is the same, whatever be the *combination* of *selected* remainders within the sequences  $s_h$  ( $1 \leq h \leq k$ ) that make the partial sum  $S_k$ . Since  $c_k$  not depends on the *combination* of *selected* remainders within the sequences  $s_h$ , the number  $c_k$  of *permitted*  $k$ -tuples within the interval  $\mathbb{I}[1, m_k]$  will not change if we choose another *selected* remainders. However, the positions of the *permitted*  $k$ -tuples along the interval  $\mathbb{I}[1, m_k]$  depends on the *combination* of *selected* remainders within the sequences  $s_h$ . That is to say, if we choose another *selected* remainders, the positions of the *permitted*  $k$ -tuples along the interval  $\mathbb{I}[1, m_k]$  will be changed. Consequently, some *permitted*  $k$ -tuples will be transfered from the Left interval  $\mathbb{I}[1, y]$  to the Right interval  $\mathbb{I}[y + 1, m_k]$  or contrariwise. In other words, the number of *permitted*  $k$ -tuples within the Left interval  $\mathbb{I}[1, y]$  (the Right interval  $\mathbb{I}[y + 1, m_k]$ ) depends on the size  $y$  (the size  $m_k - y$ ) of the interval and depends on the *combination* of *selected* remainders within the sequences  $s_h$  that make the partial sum  $S_k$ .

Let us consider now the density of *permitted*  $k$ -tuples within the intervals  $\mathbb{I}[1, m_k]$ ,  $\mathbb{I}[1, y]$  and  $\mathbb{I}[y + 1, m_k]$  within  $S_k$ . Because the number of *permitted*  $k$ -tuples within the interval  $\mathbb{I}[1, m_k]$  not depends on the *combination* of *selected* remainders within the sequences  $s_h$  that make the partial sum  $S_k$ , the density  $\delta_k$  within the interval  $\mathbb{I}[1, m_k]$  not depends on the *combination* of *selected* remainders within the sequences  $s_h$  either. But the density  $\delta_k^{\mathbb{L}}$  ( $\delta_k^{\mathbb{R}}$ ) within the Left interval  $\mathbb{I}[1, y]$  (Right interval  $\mathbb{I}[y + 1, m_k]$ ) does depend on the size  $y$  (the size  $m_k - y$ ) of the interval and does depend on the *combination* of *selected* remainders within the periods of the sequences  $s_h$  that make the partial sum  $S_k$ . From Theorem 4.4, if we take into account all the *combinations* of *selected* remainders within the sequences  $s_h$  that make the partial sum  $S_k$ , the average of  $\delta_k^{\mathbb{L}}$  within the interval  $\mathbb{I}[1, y]$  is equal to  $\delta_k$  and the average of  $\delta_k^{\mathbb{R}}$  within the interval  $\mathbb{I}[y + 1, m_k]$  is also equal to  $\delta_k$ . Therefore, we have:

$$\begin{aligned} \delta_k^{\mathbb{L}} > \delta_k &\iff \delta_k^{\mathbb{R}} < \delta_k \\ \delta_k^{\mathbb{L}} < \delta_k &\iff \delta_k^{\mathbb{R}} > \delta_k \end{aligned} \tag{6.1}$$

Now, let us consider the interval of  $k$ -tuples denoted by  $\mathbb{I}[1, y = p_k^2] = \mathbb{I}[1, p_k^2]$ , of size  $y = p_k^2$ , for every partial sum  $S_k$ . At this point, some natural questions arises: Could be possible that  $\delta_k^{\mathbb{L}} > \delta_k$  and  $\delta_k^{\mathbb{R}} < \delta_k$  for every partial sum  $S_k$  from a level  $k$  onward and for a given selection of remainders for each  $S_k$ ? Could be possible that  $\delta_k^{\mathbb{L}} < \delta_k$  and  $\delta_k^{\mathbb{R}} > \delta_k$  for every partial sum  $S_k$  from a level  $k$  onward and for a given selection of remainders for each  $S_k$ ? More specifically, could be possible that  $\delta_k^{\mathbb{L}}$  have an upper bound for all the partial sums  $S_k$  from a level  $k$  onward and for a given selection of remainders for each  $S_k$ ?

To answer these questions let us consider a given partial sum  $S_k$  and the sequences  $s_h$  ( $1 \leq h \leq k$ ) that make  $S_k$ . The density  $\delta_k^{\mathbb{L}}$  is determined by the *selected* remainders within the Left intervals  $\mathbb{I}[1, p_k^2]_h$  of the sequences  $s_h$  from

level  $h = 1$  to level  $h = k$ , and the density  $\delta_k^{\mathbb{R}}$  is determined by the *selected* remainders within the Right intervals  $\mathbb{I}[p_k^2 + 1, m_k]_h$  of the sequences  $s_h$  from level  $h = 1$  to level  $h = k$ . If our response to these questions is affirmative thus we should admit that for all the partial sums of the sequence  $\{S_k\}$  and for a given selection of remainders for each  $S_k$ , within the period  $\mathbb{I}[1, m_k]$  the *selected* remainders works in a way for the Left interval  $\mathbb{I}[1, p_k^2]$  and works in another way for the Right interval  $\mathbb{I}[p_k^2 + 1, m_k]$ . But there is not reasons to believe that it is possible, because on the one hand, for any given partial sum  $S_k$  the sequences  $s_h$  are periodic and consequently the *selected* remainders exhibits a regular pattern along each sequence  $s_h$ . On the other hand, for a given partial sum  $S_k$  the size of the interval  $\mathbb{I}[1, p_k^2]$  is not multiple of the period of any of the sequences  $s_h$  that make  $S_k$ . Besides, for  $k \rightarrow \infty$  the size of the intervals  $\mathbb{I}[1, p_k^2]$  grows up. Therefore, the response to the questions must be negative, and consequently, since from Theorem 3.4 we have  $\delta_k \rightarrow \infty$ , we make the following:

**Assumption A.** We assume that for  $k \rightarrow \infty$  we have  $\delta_k^{\mathbb{L}} \rightarrow \infty$  and  $\delta_k^{\mathbb{R}} \rightarrow \infty$ , each one fluctuating around the average  $\delta_k$  and subject to the restrictions expressed by the inequalities (6.1).

From now on we return to the notation  $\delta_k^{\mathbb{I}}$  instead of  $\delta_k^{\mathbb{L}}$ , for the *true* density of *permitted*  $k$ -tuples within the Left interval  $\mathbb{I}[1, y = p_k^2] = \mathbb{I}[1, p_k^2]$ . In the next section we formulate a probabilistic model for the *true* density  $\delta_k^{\mathbb{I}}$ .

## 6.2 Probabilistic Model for the *true* density $\delta_k^{\mathbb{I}}$

**Definition 6.10.** Collection  $\mathcal{U}_k^A$ .

Let  $s_h$  ( $1 \leq h \leq k$ ) be the sequences of unary tuples that make the partial sum  $S_k$ . Let  $\nu_k$  be the number of *combinations* of *selected* remainders within the periods of the sequences  $s_h$  ( $1 \leq h \leq k$ ). We denote by  $\mathcal{U}_k^A$  the collection of *positional patterns* within a period of  $S_k$  that come from all the *combinations* of *selected* remainders within the sequences  $s_h$ . Therefore, for a given level  $k$ , the total number of *positional patterns* is equal to  $\nu_k$ .

**Definition 6.11.** Experiment  $\varepsilon_A$ .

Let  $S_k$  ( $k \geq 3$ ) be a given partial sum. Let  $s_h$  ( $1 \leq h \leq k$ ) be the sequences of unary tuples that make the partial sum  $S_k$ . Let  $p_k$  be the characteristic prime of the partial sum  $S_k$ . Let  $m_k$  be the period of  $S_k$ . Let  $c_k$  be the number of *permitted*  $k$ -tuples within the period of  $S_k$ . Let  $\mathbb{I}[1, p_k^2]$  be the interval of size  $p_k^2$  within the partial sum  $S_k$ . We define the random experiment  $\varepsilon_A$  to be the following:

We choose randomly one *combination* of *selected* remainders within the sequences of unary tuples  $s_h$ . The choosen *combination* of *selected* remainders has one associated *positional pattern* included in the collection  $\mathcal{U}_k^A$ . Next we count the *permitted*  $k$ -tuples within the interval  $\mathbb{I}[1, p_k^2]$ . We define the random variable  $X_k^A$  as the number of *permitted*  $k$ -tuples that we found within the interval  $\mathbb{I}[1, p_k^2]$  each time we perform the experiment  $\varepsilon_A$ . We denote as  $P(X_k^A = j)$  the probability that  $X_k^A = j$  ( $j = 0, 1, 2, 3, \dots < p_k^2$ ). We denote by  $E(X_k^A)$  and  $V(X_k^A)$  respectively the expected value and the variance of  $X_k^A$ .

**Lemma 6.2.** We have  $E(X_k^A) = p_k \delta_k$ .

*Proof.* Let  $S_k$  be the partial sum of level  $k$ . Let  $s_h$  ( $1 \leq h \leq k$ ) be the sequences of unary tuples that make  $S_k$ . Let  $\bar{c}_k^{\mathbb{I}}$  be the average number of *permitted*  $k$ -tuples within the interval  $\mathbb{I}[1, p_k^2]$ , for all the *combinations* of *selected* remainders within the periods of the sequences  $s_h$ . Therefore, we have  $E(X_k^A) = \bar{c}_k^{\mathbb{I}}$ . From Theorem 4.4 the average density of *permitted*  $k$ -tuples within the interval  $\mathbb{I}[1, p_k^2]$ , for all the *combinations* of *selected* remainders within the periods of the sequences  $s_h$  is equal to  $\delta_k$ . On the other hand, within  $\mathbb{I}[1, p_k^2]$  we have  $p_k$  intervals of size  $p_k$ . Therefore, by definition we have  $E(X_k^A) = \bar{c}_k^{\mathbb{I}} = p_k \delta_k$ . ■

**Definition 6.12.** Let  $S_k$  be a given partial sum ( $k \geq 3$ ). Let  $p_k$  be the characteristic prime of the partial sum  $S_k$ . Let  $\mathbb{I}[1, p_k^2]$  be the interval of size  $p_k^2$  within the partial sum  $S_k$ . We define the random variable  $Y_k^A$  by the equation:

$$Y_k^A = \frac{X_k^A}{p_k}$$

We denote by  $E(Y_k^A)$  and  $V(Y_k^A)$  respectively the expected value and the variance of  $Y_k^A$ .

**Lemma 6.3.** Let  $S_k$  be a given partial sum ( $k \geq 3$ ) and let  $\delta_k$  be the average density of *permitted*  $k$ -tuples within the period of the partial sum  $S_k$ . Let  $E(Y_k^A)$  be the expected value of the random variable  $Y_k^A$ .

We have  $E(Y_k^A) = \delta_k$ .

*Proof.* By definition we have  $E(Y_k^A) = E(X_k^A/p_k) = 1/p_k E(X_k^A)$ . Therefore, using Lemma 6.2 we have  $E(Y_k^A) = 1/p_k (p_k \delta_k) = \delta_k$ . ■

**Remark 6.1.** The random variable  $Y_k^A$  denotes the average number of *permitted*  $k$ -tuples for the  $p_k$  subintervals of size  $p_k$  that there are within  $\mathbb{I}[1, p_k^2]$ , each time we perform the experiment  $\varepsilon_A$ , and thus, by definition, we have  $Y_k^A = \delta_k^{\mathbb{I}}$ .

## 6.3 Hypergeometric Model

**Assumption B.** Within a given partial sum  $S_k$ , the  $c_k$  *permitted*  $k$ -tuples can be placed in any position along the period of  $S_k$ .

**Remark 6.2.** The Assumption B represents an ideal situation where the  $c_k$  *permitted*  $k$ -tuples can be located arbitrarily at any position along the period of  $S_k$  and it is only valid to define the **Hypergeometric Model**.

**Definition 6.13.** Collection  $\mathcal{U}_k^B$ .

Let  $m_k$  denote the period of  $S_k$ . We denote by  $\mathcal{U}_k^B$  the collection of *positional patterns* that come from all the ways that the  $c_k$  *permitted*  $k$ -tuples could be scattered over the period of  $S_k$ , taking into account the Assumption B. Therefore, for a given level  $k$ , the number of *positional patterns* that comply with the Assumption B is equal to:

$$|\mathcal{U}_k^B| = \binom{m_k}{c_k}$$

**Definition 6.14.** Experiment  $\varepsilon_B$ .

Let  $S_k$  ( $k \geq 3$ ) be a given partial sum. Let  $p_k$  be the characteristic prime of the partial sum  $S_k$ . Let  $m_k$  be the period of  $S_k$ . Let  $c_k$  be the number of *permitted*  $k$ -tuples within the period of  $S_k$ . Let  $\mathbb{I}[1, p_k^2]$  be the interval of size  $p_k^2$  within the partial sum  $S_k$ . We define the random experiment  $\varepsilon_B$  to be the following:

First, taking into account the Assumption B, we spread randomly the  $c_k$  *permitted*  $k$ -tuples along the  $m_k$  positions that we have within a period of  $S_k$ . Next we count the *permitted*  $k$ -tuples within the interval  $\mathbb{I}[1, p_k^2]$ . We define the random variable  $X_k^B$  as the number of *permitted*  $k$ -tuples that we found within the interval  $\mathbb{I}[1, p_k^2]$  each time we perform the experiment  $\varepsilon_B$ . The probability that  $X_k^B = j$  will be:

$$P(X_k^B = j) = \frac{\binom{c_k}{j} \binom{m_k - c_k}{p_k^2 - j}}{\binom{m_k}{p_k^2}}, \quad j = 0, 1, 2, 3, \dots \leq p_k^2$$

that is to say,  $X_k^B$  has an Hypergeometric Distribution with parameters  $(p_k^2, c_k, m_k)$ . Denoting by  $E(X_k^B)$  and  $V(X_k^B)$  respectively the expected value and the variance of  $X_k^B$ , we have:

$$E(X_k^B) = p_k^2 \frac{c_k}{m_k} \quad V(X_k^B) = p_k^2 \frac{c_k}{m_k} \left(1 - \frac{c_k}{m_k}\right) \frac{m_k - p_k^2}{m_k - 1}$$

**Lemma 6.4.** Let  $S_k$  be a given partial sum. Let  $p_k$  be the characteristic prime of the partial sum  $S_k$ . Let  $m_k$  be the period of  $S_k$ . Let  $c_k$  be the number of *permitted*  $k$ -tuples within the period of  $S_k$ . We have the identity:

$$\frac{c_k}{m_k} = \frac{\delta_k}{p_k}$$

*Proof.* Within a period of the partial sum  $S_k$  the number of intervals of size  $p_k$  is equal to  $m_k/p_k$ . Therefore, by definition we have:

$$\frac{c_k}{m_k} = \frac{1}{p_k} \frac{c_k}{m_k/p_k} = \frac{\delta_k}{p_k}$$

■

**Lemma 6.5.** We have  $E(X_k^B) = p_k \delta_k$ .

*Proof.* Using Lemma 6.4 we have:

$$E(X_k^B) = p_k^2 \frac{c_k}{m_k} = p_k^2 \frac{\delta_k}{p_k} = p_k \delta_k$$

■

**Lemma 6.6.** For  $k \rightarrow \infty$  we have  $V(X_k^B) \rightarrow \infty$ .

*Proof.* By definition we have:

$$V(X_k^B) = p_k^2 \frac{c_k}{m_k} \left(1 - \frac{c_k}{m_k}\right) \frac{m_k - p_k^2}{m_k - 1}$$

Using Lemma 6.4 we have:

$$V(X_k^B) = p_k^2 \frac{\delta_k}{p_k} \left(1 - \frac{c_k}{m_k}\right) \frac{m_k - p_k^2}{m_k - 1} = p_k \delta_k \left(1 - \frac{c_k}{m_k}\right) \frac{m_k - p_k^2}{m_k - 1}$$

From Lemma 2.3 we have  $c_k = o(m_k)$ , from Corollary 2.5 we have  $p_k^2 = o(m_k)$  and from Theorem 3.4 we have  $\delta_k \rightarrow \infty$ . Thus we have  $V(X_k^B) \rightarrow \infty$ , as desired. ■

**Definition 6.15.** Let  $S_k$  be a given partial sum ( $k \geq 3$ ). Let  $p_k$  be the characteristic prime of the partial sum  $S_k$ . Let  $\mathbb{I}[1, p_k^2]$  be the interval of size  $p_k^2$  within the partial sum  $S_k$ . We define the random variable  $Y_k^B$  by the equation:

$$Y_k^B = \frac{X_k^B}{p_k}$$

The random variable  $Y_k^B$  denotes the average number of *permitted*  $k$ -tuples for the  $p_k$  subintervals of size  $p_k$  that there are within  $\mathbb{I}[1, p_k^2]$ , each time we perform the experiment  $\varepsilon_B$ . We denote by  $E(Y_k^B)$  and  $V(Y_k^B)$  respectively the expected value and the variance of  $Y_k^B$ .

**Lemma 6.7.** Let  $S_k$  be a given partial sum ( $k \geq 3$ ) and let  $\delta_k$  be the average density of permitted  $k$ -tuples within the period of the partial sum  $S_k$ . We have  $E(Y_k^B) = \delta_k$ .

*Proof.* By definition we have  $E(Y_k^B) = E(X_k^B/p_k) = 1/p_k E(X_k^B)$ . Therefore, using Lemma 6.5 we have  $E(Y_k^B) = 1/p_k (p_k \delta_k) = \delta_k$ . ■

**Lemma 6.8.** We have:

$$V(Y_k^B) = \frac{c_k}{m_k} \left(1 - \frac{c_k}{m_k}\right) \frac{m_k - p_k^2}{m_k - 1}$$

*Proof.* By definition we have:

$$V(Y_k^B) = V\left(\frac{X_k^B}{p_k}\right) = \frac{1}{p_k^2} V(X_k^B) = \frac{c_k}{m_k} \left(1 - \frac{c_k}{m_k}\right) \frac{m_k - p_k^2}{m_k - 1}$$

**Lemma 6.9.** For level  $k > 4$  we have  $0 \leq X_k^B \leq p_k^2$  and  $0 \leq Y_k^B \leq p_k$ .

*Proof.* Let  $S_k$  be a given partial sum ( $k > 4$ ). From Lemma 2.4 we have  $p_k^2 = o(c_k)$ . Thus it is easy to check that for level  $k > 4$  we have  $c_k > p_k^2$ . Consequently, by definition, the random variable  $X_k^B$  range from 0 (There does not exist any *permitted*  $k$ -tuple within  $\mathbb{I}[1, p_k^2]$ ) to  $p_k^2$  (There exist  $p_k^2$  *permitted*  $k$ -tuples within  $\mathbb{I}[1, p_k^2]$ ). Thus, by definition, the random variable  $Y_k^B$  range from 0 to  $p_k$ . ■

## 6.4 A Lower Confidence Limit for $Y_k^B$

Let us consider the intervals  $\mathbb{I}[1, m_k]$  within the sequence of partial sums  $S_k$  (The first period of every partial sum  $S_k$ ) as a sequence of dichotomous finite populations indexed by  $k$ , where we have the population of *permitted*  $k$ -tuples and the population of *prohibited*  $k$ -tuples. For each level  $k$ , by definition, the random variable  $X_k^B$  represents the number of *permitted*  $k$ -tuples within the interval  $\mathbb{I}[1, p_k^2]$  and it has an Hypergeometric distribution with parameters  $(p_k^2, c_k, m_k)$ . Thus we have the population of *permitted*  $k$ -tuples fraction given by  $c_k/m_k$  and the sampling fraction given by  $p_k^2/m_k$ . Since from Lemma 6.6 we have  $V(X_k^B) \rightarrow \infty$  as  $k \rightarrow \infty$ , it follows from a result of Lahiri and Chatterjee ([9], Theorem 2.1) that we can use the Normal distribution function to approximate the distribution of  $X_k^B$ . Now, let  $L_k^{XB} = E(X_k^B) - 3.09\sqrt{V(X_k^B)}$  and suppose that for a level  $k$  large enough the distribution of  $X_k^B$  becomes very close to the Normal distribution. It follows that in this case would be  $P(X_k^B \leq L_k^{XB}) \approx \alpha = 0.001$ . However, from Lemma 2.3 we have  $c_k = o(m_k)$ , and it is easy to check that for  $k \geq 4$  we have  $c_k < m_k/2$ . Therefore, since the distribution function of  $X_k^B$  is Hypergeometric, it must be positively asymmetrical. We can see in the Table 6.1, last column, that from level  $k = 4$  to level  $k = 6$ , the values of  $L_k^{XB}$  are negatives, because of the asymmetry of the distribution of  $X_k^B$ . Consequently, we assume that as the level grows up from  $k = 7$  onward,  $L_k^{XB}$  grows up together with  $P(X_k^B \leq L_k^{XB})$ , which holds bounded by  $\alpha = 0.001$ . In other words, we assume for  $k \geq 7$  that  $P(X_k^B \leq L_k^{XB}) = \alpha$ ,  $\alpha \leq 0.001$ . Thus, since  $Y_k^B = X_k^B/p_k$  by definition, it makes sense to define a lower confidence limit for  $Y_k^B$ :

**Definition 6.16.** Let  $S_k$  be a given partial sum, where  $k \geq 7$ . Let  $E(Y_k^B)$  and  $V(Y_k^B)$  be the expected value and the variance of the random variable  $Y_k^B$  respectively. We define the  $100(1 - \alpha)\%$  Lower Confidence Limit for the random variable  $Y_k^B$ , where  $\alpha \leq 0.001$ , by the equation:

$$L_k^{YB} = E(Y_k^B) - 3.09 \sqrt{V(Y_k^B)}$$

Table 6.1:

$k$	$p_k$	$p_k^2$	$m_k$	$c_k$	$\delta_k$	$E(X_k^B)$	$V(X_k^B)$	$L_k^{XB}$
4	7	49	210	15	0.500	3.500	2.50	-1.39
5	11	121	2310	135	0.643	7.071	6.31	-0.69
6	13	169	30030	1485	0.643	8.357	7.90	-0.33
7	17	289	510510	22275	0.742	12.610	12.05	1.88
8	19	361	9699690	378675	0.742	14.093	13.54	2.72
9	23	529	223092870	7952175	0.820	18.856	18.18	5.68
10	29	841	6469693230	214708725	0.962	27.910	26.98	11.86
11	31	961	-	-	0.962	29.835	28.91	13.22
12	37	1369	-	-	1.087	40.204	39.02	20.90
13	41	1681	-	-	1.145	46.959	45.65	26.08
			-	-				.

From level  $k = 4$  to level  $k = 13$ , Table 6.1 shows the following:

**Column 1** Level  $k$ .

**Column 2** Characteristic prime  $p_k$ .

**Column 3**  $p_k^2$ .

**Column 4** The period  $m_k$ .

**Column 5**  $c_k$  (Number of *permitted*  $k$ -tuples within the period).

**Column 6**  $\delta_k$  (Average density of *permitted*  $k$ -tuples).

**Column 7**  $E(X_k^B) = p_k \delta_k$  (Average number of *permitted*  $k$ -tuples within the interval  $\mathbb{I}[1, p_k^2]$ ).

**Column 8**  $V(X_k^B) = p_k^2 \frac{c_k}{m_k} \left(1 - \frac{c_k}{m_k}\right) \frac{m_k - p_k^2}{m_k - 1}$ .

**Column 9**  $L_k^{XB} = E(X_k^B) - 3.09\sqrt{V(X_k^B)}$ .

**Lemma 6.10.** For  $k \rightarrow \infty$  we have  $L_k^{YB} \rightarrow \infty$ .

*Proof.* From Lemma 6.7 and Lemma 6.8 we have:

$$L_k^{YB} = E(Y_k^B) - 3.09 \sqrt{V(Y_k^B)} = \delta_k - 3.09 \sqrt{\frac{c_k}{m_k} \left(1 - \frac{c_k}{m_k}\right) \frac{m_k - p_k^2}{m_k - 1}}$$

From Proposition 2.3 we have  $c_k = o(m_k)$  and from Corollary 2.5 we have  $p_k^2 = o(m_k)$ . On the other hand, from Theorem 3.4 we have  $\delta_k \rightarrow \infty$ . It follows that  $L_k^{YB} \rightarrow \infty$ . ■

**Remark 6.3.** It is easy to check that  $L_k^{YB}$  grows up monotonically from  $k = 7$  onward.

## 6.5 Comparison of the Models

Let us compare the distribution of the random variable  $X_k^A$  with the distribution of the random variable  $X_k^B$  (Hypergeometric). On the one hand, from Lemma 6.2 and Lemma 6.5, we have that the expected value of  $X_k^A$  and  $X_k^B$  is the same. On the other hand, using the Construction Procedure (Proposition 2.6), if we take  $m_k/6$  periods of the partial sum  $S_2$  and then juxtapose the remainders of the sequences  $s_h$  from  $h = 3$  to  $h = k$  we get a period of the partial sum  $S_k$  ( $k > 4$ ). At level  $h = 2$  the distance between consecutive *permitted* 2-tuples is equal to 6, but as we add the sequences  $s_h$  the distance between some consecutive *permitted*  $h$ -tuples grows up to a multiple of 6 (see Proposition 6.1). Without loss of generality, the *positional pattern* where all the  $k$ -tuples within the interval  $\mathbb{I}[1, p_k^2]$  are *permitted*  $k$ -tuples is not included within the collection  $\mathcal{U}_k^A$ , but it is included within the collection  $\mathcal{U}_k^B$ , because by Assumption B, the collection  $\mathcal{U}_k^B$  includes the *positional patterns* that outcomes from all the ways the *permitted*  $k$ -tuples could be spread along the period of  $S_k$ , regardless of the distance between consecutive *permitted*  $k$ -tuples. The same is true for others *positional patterns* included within the collection  $\mathcal{U}_k^B$ , that contribute to the tails on the left and right sides of the distribution of the random variable  $X_k^B$ , but are not included within the collection  $\mathcal{U}_k^A$  and do not contribute to the tails of the distribution of the random variable  $X_k^A$ .

**Remark 6.4.** Note that the *positional patterns* included within the collection  $\mathcal{U}_k^A$  are the ones where all the distances between consecutive *permitted*  $k$ -tuples are close to the average distance ( $m_k/c_k$ ), while the Assumption B allows for the *positional patterns* included within the collection  $\mathcal{U}_k^B$  such that the distances between consecutive *permitted*  $k$ -tuples could be far away from the average distance.

Now, let us compare the distribution of the random variable  $Y_k^A = X_k^A/p_k$  with the distribution of the random variable  $Y_k^B = X_k^B/p_k$ . Once again, from Lemma 6.3 and Lemma 6.7 we have  $E(Y_k^A) = E(Y_k^B) = \delta_k$ . On the other hand, since the random variable  $X_k^A$  can not reach the larger values (up to  $p_k^2$ ) of the random variable  $X_k^B$ , the random variable  $Y_k^A$  can not reach the larger values (up to  $p_k$ ) of the random variable  $Y_k^B$  (see Lemma 6.9). In the same way, since the random variable  $X_k^A$  can not reach the smaller values (from 0) of the random variable  $X_k^B$ , the random variable  $Y_k^A$  can not reach the smaller values (from 0) of the random variable  $Y_k^B$  (see Lemma 6.9).

## 6.6 Calculating the number of *permitted* $k$ -tuples within $\mathbb{I}[1, p_k^2]$

Finally, we prove that for  $k \geq 13$  the number of *permitted*  $k$ -tuples within the interval  $\mathbb{I}[1, p_k^2]$  is greater than 3.

**Remark 6.5.** For the computing of the densities of *permitted*  $k$ -tuples within a given interval we round the results to 3 digits to the right of the decimal point.

Let  $\epsilon > 0$ ,  $\epsilon \in \mathbb{R}$  be a small enough number. Let  $S_h$  be the partial sums from level  $h = 1$  to level  $h = k$ ,  $k > 4$ . Let  $\mathbb{I}[1, p_k^2]_h$  be the interval of  $h$ -tuples of size  $p_k^2$ , within every partial sum  $S_h$  ( $1 \leq h \leq k$ ). Let  $\delta_h$  be the average density of *permitted*  $h$ -tuples within every interval  $\mathbb{I}[1, p_k^2]_h$  ( $1 \leq h \leq k$ ). Let  $\delta_h^\mathbb{I}$  be the *true* density of *permitted*  $h$ -tuples within every interval  $\mathbb{I}[1, p_k^2]_h$  ( $1 \leq h \leq k$ ).

For level  $h = 4$ , let us consider the *true* density  $\delta_4^\mathbb{I}$  within the interval  $\mathbb{I}[1, p_k^2]_{h=4}$ , in the partial sum  $S_{h=4} = S_4$ . As we increase the level  $k$ , the size of the interval  $\mathbb{I}[1, p_k^2]_{h=4}$  within the partial sum  $S_4$  grows up. Therefore, from Proposition 5.2, we can reach a level  $k$  large enough such that from this level onward we have:

$$(\delta_4 - \epsilon) \leq \delta_4^\mathbb{I} \leq (\delta_4 + \epsilon) \tag{6.2}$$

On the other hand, for a given level  $k$ , as we go up from level  $h = 4$  to level  $h = k$ , the *selected* remainders of the sequences  $s_h$  remove *permitted*  $h$ -tuples within the intervals  $\mathbb{I}[1, p_k^2]_h$ . However, the average density  $\delta_h$  within the interval  $\mathbb{I}[1, p_k^2]_h$  grows up, because to compute the average density  $\delta_h$  we count the number of *permitted*  $h$ -tuples



within subintervals of size  $p_h$ , which grows up from level  $h = 4$  to level  $h = k$ , compensating in excess for the *permitted*  $h$ -tuples removed (see Proposition 3.2 and Corollary 3.3). Consequently, denoting by  $\bar{\Delta}(p_4 \rightarrow p_k)$  the increment of the average density from level  $h = 4$  to level  $h = k$ , we have  $\bar{\Delta}(p_4 \rightarrow p_k) > 0$ .

Let us now consider the case of the *true* density  $\delta_k^{\mathbb{I}}$ . From Corollary 3.3, if  $p_h \rightarrow p_{h+1}$  is a level transition of *order* 2 we have  $\bar{\Delta}(p_h \rightarrow p_{h+1}) = 0$ , that is to say, the average density  $\delta_h$  not change at the level transitions of *order* 2. Thus, the *true* increment for a given transition of *order* 2 could be  $\Delta(p_h \rightarrow p_{h+1}) > 0$ ,  $\Delta(p_h \rightarrow p_{h+1}) = 0$  or  $\Delta(p_h \rightarrow p_{h+1}) < 0$ . If  $p_h \rightarrow p_{h+1}$  is a level transition of *order*  $> 2$  we have  $\bar{\Delta}(p_h \rightarrow p_{h+1}) > 0$ , that is to say, the average density  $\delta_h$  grows up at the level transitions of *order*  $> 2$ . It is known that for  $h \geq 4$  ( $p_h \geq 7$ ) can not exist two consecutive level transitions of *order* 2. Therefore, from  $h = 4$  to  $h = k$  ( $k > 4$ ), for each level transition of *order* 2 we have one or more level transitions of *order*  $> 2$ . Thus, denoting by  $\Delta(p_4 \rightarrow p_k)$  the increment of the *true* density from level  $h = 4$  to level  $h = k$ , it seems reasonable to assume  $\Delta(p_4 \rightarrow p_k) > 0$ . However, one can ask if it is possible, for a given *combination* of *selected* remainders within the periods of the sequences  $s_h$  ( $1 \leq h \leq k$ ) that  $\Delta(p_4 \rightarrow p_k) \leq 0$ . Indeed, for a low level  $k > 4$ , such that between the level transitions from  $h = 4$  to  $h = k$  predominate the transitions of *order* 2 over the transitions of *order*  $> 2$  and for a given *combination* of *selected* remainders, the answer could be "yes". But, by Assumption A there must exist a level  $k$  large enough such that from this level onward we have  $Y_k^A = \delta_k^{\mathbb{I}} > (\delta_4 + \epsilon) \implies \Delta(p_4 \rightarrow p_k) > 0$ , regardless of the *combination* of *selected* remainders within the periods of the sequences  $s_h$  ( $1 \leq h \leq k$ ) that make  $S_k$ . In this case, from Eq. (6.2), we have  $(\delta_4 - \epsilon) \leq \delta_4^{\mathbb{I}} \leq (\delta_4 + \epsilon) \wedge \Delta(p_4 \rightarrow p_k) > 0 \implies \delta_k^{\mathbb{I}} = \delta_4^{\mathbb{I}} + \Delta(p_4 \rightarrow p_k) > \delta_4^{\mathbb{I}} > (\delta_4 - \epsilon)$ .

At this point, a natural question arises: How can we find this level  $k$  large enough such that from this level onward we have  $(\delta_4 - \epsilon) \leq \delta_4^{\mathbb{I}} \leq (\delta_4 + \epsilon)$  and  $\Delta(p_4 \rightarrow p_k) > 0$  at the same time?. To answer this question let us consider the level  $k$  where  $(\delta_4 - \epsilon) \leq \delta_4^{\mathbb{I}} \leq (\delta_4 + \epsilon)$  and the Lower Confidence Limit  $L_k^{YB}$  becomes greater than  $(\delta_4 + \epsilon)$ . By definition, from this level  $k$  onward we have  $P(Y_k^B > L_k^{YB}) = 1 - \alpha$ ,  $\alpha \leq 0.001$ . Thus, from what we discussed in Section 6.5, from this level  $k$  onward we assume  $Y_k^A = \delta_k^{\mathbb{I}} > L_k^{YB}$ . In other words, we assume that for the level  $k$  where the Lower Confidence Limit  $L_k^{YB}$  becomes greater than  $(\delta_4 + \epsilon)$  onward, the likelihood of founding values of the random variable  $Y_k^A = \delta_k^{\mathbb{I}}$  less than or equal to  $L_k^{YB}$  is null. It follows that from this level onward we have  $(\delta_4 - \epsilon) \leq \delta_4^{\mathbb{I}} \leq (\delta_4 + \epsilon)$  and  $\Delta(p_4 \rightarrow p_k) > 0$  simultaneously.

**Lemma 6.11.** *Let  $S_k$  be a partial sum of the series  $\{S_k\}$ , where  $k \geq 13$  ( $p_k \geq 41$ ). Let  $\delta_k^{\mathbb{I}}$  denote the true density of permitted  $k$ -tuples within the interval  $\mathbb{I}[1, p_k^2]$  of the partial sum  $S_k$ . We have  $\delta_k^{\mathbb{I}} > 0.430$ .*

*Proof.* Let  $\epsilon = 0.070$ . For level  $h = 4$ , let us consider the interval  $\mathbb{I}[1, p_k^2]_{h=4}$ , within the partial sum  $S_{h=4} = S_4$ . For level  $k = 13$  we have  $p_k^2 = p_{13}^2 = 1681$ . From Proposition 3.1, the average density within  $\mathbb{I}[1, p_k^2]_{h=4}$  is equal to  $\delta_4 = 0.500$ . Consequently, for the interval  $\mathbb{I}[1, p_{13}^2]_{h=4}$ , using Proposition 5.3 we have  $(\delta_4 - \epsilon) \leq \delta_4^{\mathbb{I}} \leq (\delta_4 + \epsilon)$ . It is easy to check, using Proposition 5.3, that for level  $k > 13$  it also holds  $(\delta_4 - \epsilon) \leq \delta_4^{\mathbb{I}} \leq (\delta_4 + \epsilon)$ .

On the other hand, inspecting Table 6.2 and Table 6.3 we find that for level  $k = 13$  we have  $L_{13}^{YB} > (\delta_4 + \epsilon)$ , and from Lemma 6.10 and Remark 6.3, we have that the inequality  $L_k^{YB} > \delta_4 + \epsilon$  also holds for level  $k > 13$ . Therefore, by assumption, for level  $k \geq 13$  we have  $Y_k^A = \delta_k^{\mathbb{I}} > L_k^{YB} > (\delta_4 + \epsilon) > (\delta_4 - \epsilon) = 0.430$ , as desired. ■

**Remark 6.6.** Note that between level  $h = 4$  ( $p_h = p_4 = 7$ ) and level  $h = k = 13$  ( $p_k = p_{13} = 41$ ) there are 6 level transitions of *order*  $> 2$ :

$$\begin{array}{ll} 7 & \longrightarrow 11 \\ 13 & \longrightarrow 17 \\ 19 & \longrightarrow 23 \\ 23 & \longrightarrow 29 \\ 31 & \longrightarrow 37 \\ 37 & \longrightarrow 41 \end{array}$$

and only 3 level transitions of *order* 2:

$$\begin{array}{ll} 11 & \longrightarrow 13 \\ 17 & \longrightarrow 19 \\ 29 & \longrightarrow 31 \end{array}$$

**Proposition 6.12.** *Let  $c_k^{\mathbb{I}}$  denote the number of permitted  $k$ -tuples within the interval  $\mathbb{I}[1, p_k^2]$  of the partial sum  $S_k$ . For level  $k \geq 13$  ( $p_k \geq 41$ ) we have  $c_k^{\mathbb{I}} > 3$ .*

*Proof.* From Lemma 6.11, for level  $k \geq 13$  ( $p_k \geq 41$ ), we have  $\delta_k^{\mathbb{I}} > 0.430 \implies c_k^{\mathbb{I}} = p_k \delta_k^{\mathbb{I}} > 41 \times 0.430 > 3$ . ■

Table 6.2: Bounds for  $\delta_4^{\mathbb{I}}$  from Proposition 5.3

$k$	$p_k$	$p_k^2$	Lower Bound	Upper Bound
4	7	49	-	-
5	11	121	-	-
6	13	169	-	-
7	17	289	0.2506	0.9953
8	19	361	0.2506	0.9953
9	23	529	0.3339	0.7482
10	29	841	0.4004	0.6243
11	31	961	0.4004	0.6243
12	37	1369	0.4289	0.5829
13	41	1681	0.4447	0.5622
14	43	1849	0.4447	0.5622
15	47	2209	0.4547	0.5497
16	53	2809	0.4644	0.5383
17	59	3481	0.4707	0.5311
18	61	3721	0.4723	0.5293
.	.	.	.	.

Table 6.2 shows the following:

**Column 1** Level  $k$

**Column 2** Characteristic prime  $p_k$

**Column 3**  $p_k^2$

**Column 4** Lower Bound for  $\delta_4^{\mathbb{I}}$

**Column 5** Upper Bound for  $\delta_4^{\mathbb{I}}$

Table 6.3: Lower Confidence Limit for  $Y_k^B$

$k$	$p_k$	$E(Y_k^B)$	$V(Y_k^B)$	$L_k^{YB}$
4	7	0.5000	0.2260	-0.1985
5	11	0.6429	0.2284	-0.0629
6	13	0.6429	0.2162	-0.0252
7	17	0.7418	0.2042	0.1107
8	19	0.7418	0.1937	0.1433
9	23	0.8198	0.1854	0.2469
10	29	0.9624	0.1791	0.4089
11	31	0.9624	0.1734	0.4265
12	37	1.0866	0.1688	0.5649
13	41	1.1453	0.1648	0.6361
14	43	1.1453	0.1610	0.6478
15	47	1.1986	0.1576	0.7115
16	53	1.3006	0.1547	0.8225
17	59	1.3988	0.1521	0.9287
18	61	1.3988	0.1497	0.9363
.	.	.	.	.

Table 6.3 shows the following:

**Column 1** Level  $k$

**Column 2** Characteristic prime  $p_k$

**Column 3**  $E(Y_k^B) = \delta_k$  (Average density of *permitted*  $k$ -tuples)

**Column 4**  $V(Y_k^B)$

**Column 5**  $L_k^{YB} = E(Y_k^B) - 3.09 \sqrt{V(Y_k^B)}$  (Lower Confidence Limit for  $Y_k^B$ ).

## Chapter 7

# The Fundamental Theorem and the permitted $k$ -tuples

In Chapter 2 we first defined the series  $\sum s_k$  and then we defined the *selected* remainders in the sequences of unary tuples. In this chapter we denote the series as  $\sum s_k^a$ , to indicate that it is associated to an even number  $a$  by the *selected* remainders in the sequences of unary tuples  $s_k^a$  that make the partial sums  $S_k^a$ .

**Remark 7.1.** Given an even number  $a$  such that is not twice a prime, it is established the greatest prime  $p_k < \sqrt{a}$  and consequently it is also established the number  $k$  of primes less than  $\sqrt{a}$ . Therefore, it is established the level  $k$  of the partial sum  $S_k^a$ .

Let  $a \geq 1682$  be an even number such that  $a$  is not twice a prime. Let  $p_1, p_2, p_3, \dots, p_k$  be the ordered set of  $k$  primes less than  $\sqrt{a}$ . Since  $a \geq 1682 \implies \sqrt{a} > 41$ , we have  $k \geq 13$  ( $p_k \geq 41$ ).

**Definition 7.1.** Let  $\{b_1, b_2, b_3, \dots, b_k\}$  be the ordered set of the remainders of dividing  $a$  by  $p_1, p_2, p_3, \dots, p_k$ . We define the series associated to  $a$ , denoted by  $\sum s_k^a$ , to be the sequence of partial sums  $\{S_k^a\}$ , such that in the sequences of unary tuples  $s_h^a$  ( $1 \leq h \leq k$ ) that make the partial sum  $S_k^a$  are applied the following restrictions:

Restriction 1: Within every period of size  $p_h$  of the sequence  $s_h^a$  ( $1 \leq h \leq k$ ), the remainder 0 is *selected*.

Restriction 2: Within every period of size  $p_h$  of the sequence  $s_h^a$  ( $1 \leq h \leq k$ ), the remainder  $b_h$  is *selected*.

**Remark 7.2.** Every period in the sequence of unary tuples  $s_h^a$  ( $1 \leq h \leq k$ ) are made with the remainders of dividing  $n$  by  $p_h$ . If a remainder is equal to 0 is always a *selected* remainder. If a remainder is equal to  $b_h$  is also a *selected* remainder. If  $a$  is divisible by  $p_h$ , then  $b_h = 0$  and therefore in every period  $p_h$  of  $s_h^a$  there will be only one *selected* remainder.

The next theorem allow us to identify, for a given even number  $a$ , the primes  $p$  such that  $a - p$  is also a prime:

**Theorem 7.1.** *The Fundamental Theorem*

Let  $\mathbb{C}$  be the set of primes less than  $\sqrt{a}$ . Let  $q \in \mathbb{C}$ . Let  $p$  be a prime less than  $a$ . Therefore if  $p \not\equiv a \pmod{q}$  for every  $q \in \mathbb{C}$ , then  $a - p$  is a prime or  $a - p = 1$ .

*Proof.* Assume that, on the contrary,  $c = a - p$  is a composite number. Since  $c$  is less than  $a$ , then  $\sqrt{c} < \sqrt{a}$  and consequently the primes factors of  $c$  less than  $\sqrt{c}$  are in  $\mathbb{C}$ . Therefore it must exist some natural number  $m$  such that  $a - p = c = m q$ , where  $q \in \mathbb{C}$ . This means that  $p \equiv a \pmod{q}$ , but this contradicts the hypothesis. Therefore  $a - p$  must be a prime number or  $a - p = 1$ . ■

The next two lemmas prove that the indices  $n$  ( $1 \leq n < a$ ) of the *permitted*  $k$ -tuples are the numbers that comply with the hypothesis of the Fundamental Theorem.

**Lemma 7.2.** *In the partial sum  $S_k^a$  associated to  $a$ , if  $n < a$  denotes the index of an permitted  $k$ -tuple, then  $n = 1 \vee n$  is a prime .*

*Proof.* In the sequences of unary tuples  $s_h^a$  ( $1 \leq h \leq k$ ) that make the partial sum  $S_k^a$  associated to  $a$ , if a remainder is equal to 0 then is a *selected* remainder. Then, by definition, a *permitted*  $k$ -tuple in the partial sum  $S_k^a$  has not any remainder equal to 0. This means that  $n$  is not divisible by any of the  $p_1, p_2, p_3, \dots, p_k$  primes. Since  $n < a \implies \sqrt{n} < \sqrt{a} \implies n$  is not divisible by any of the primes less than  $\sqrt{n}$ . Therefore  $n = 1 \vee n$  is a prime . ■

**Lemma 7.3.** *In the partial sum  $S_k^a$ , if  $n$  ( $1 < n < a$ ) denotes the index of a permitted  $k$ -tuple, then  $a - n$  is a prime or  $a - n = 1$ .*

*Proof.* Let  $\{b_1, b_2, b_3, \dots, b_k\}$  be the ordered set of the remainders of dividing  $a$  by  $p_1, p_2, p_3, \dots, p_k$ . In the sequences  $s_h^a$  ( $1 \leq h \leq k$ ) of unary tuples that make the partial sum  $S_k^a$ , by definition, if the remainder  $r_h$  is equal to  $b_h$  then is a *selected* remainder. Consequently, a *permitted*  $k$ -tuple in the partial sum  $S_k^a$  can not have a remainder equal to any  $b_h \in \{b_1, b_2, b_3, \dots, b_k\}$ . Therefore  $r_h \neq b_h \implies n \not\equiv a \pmod{p_h}$  for every prime  $p_h < \sqrt{a}$ . Since  $1 < n < a$  is the index of a *permitted*  $k$ -tuple, from Lemma 7.2,  $n$  is a prime. Consequently, from Theorem 7.1,  $a - n$  is a prime or  $a - n = 1$ . ■

**Lemma 7.4.** *Let  $a \geq 1682$  be an even number such that  $a$  is not twice a prime. Let  $S_k^a$  be the partial sum of the series  $\sum s_k^a$  associated to  $a$ . Let  $c$  be the number of permitted  $k$ -tuples within the interval  $\mathbb{I}[1, p_k^2]$  of  $S_k^a$ . We have:  $c > 3$ .*

*Proof.* By definition, the sequences  $s_h^a$  that make a given partial sum  $S_k^a$ , for  $1 < h \leq k$ , can have one or two *selected* remainders in every period. But the sequences  $s_h$  that make a partial sum  $S_k$ , by definition, for  $1 < h \leq k$  have always two *selected* remainders in every period. From Proposition 6.12, the number of *permitted*  $k$ -tuples within the interval  $\mathbb{I}[1, p_k^2]$  of  $S_k$ , for  $k \geq 13$  ( $p_k \geq 41$ ), is  $> 3$ . Therefore, for a given partial sum  $S_k^a$ , for  $k \geq 13$  ( $p_k \geq 41$ ) we assume  $c > 3$  either. ■

**Theorem 7.5.** *For  $a \geq 1682$  we have at least 2 primes such that the sum is  $a$ .*

*Proof.* From Lemma 7.4, for  $a \geq 1682$ , within the interval  $\mathbb{I}[1, p_k^2]$  of  $S_k^a$  there exists at least 3 *permitted*  $k$ -tuples. Consequently, in the partial sum  $S_k^a$ , inside the interval  $\mathbb{I}[1, a]$  it must exist at least 3 *permitted*  $k$ -tuples. From Lemma 7.2, in the partial sum  $S_k^a$ , if  $n < a$  denotes the index of one *permitted*  $k$ -tuple, then  $n = 1 \vee n$  is a prime. Therefore, it must exist at least 2 *permitted*  $k$ -tuples, within the interval  $\mathbb{I}[1, a]$  of the partial sum  $S_k^a$ , such that their indices are primes. From Lemma 7.3 if  $1 < n < a$  denotes the index of one *permitted*  $k$ -tuple, then  $a - n = 1 \vee a - n$  is a prime. Therefore it must exist at least one  $n$  prime such that  $d = a - n$  is also a prime  $\implies n + d = n + (a - n) = a$  and the conclusion is reached that  $a$  can be expressed as the sum of two primes. ■

The result obtained in Theorem 7.5 is sufficient to prove our Main Proposition (Proposition 1.1), as desired.

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